NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS | WWW.NCTM.ORG

MATHEMATICS teacher

APRIL 2013

galileo, gauss, and the green monster

// building sinusoids

// geometric reasoning

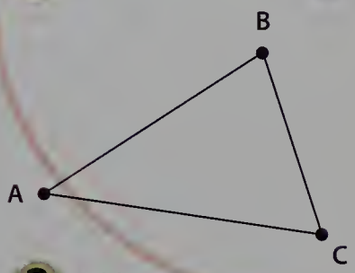
Marygrove College Library
8425 West McNichols Road
Detroit, MI 48221

Tired of using the wrong drawing tool for your math documents?

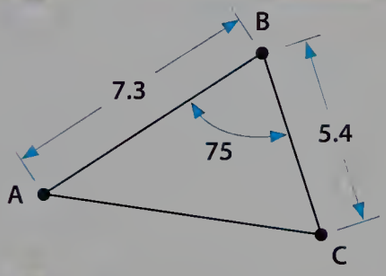
Now you can create high quality math diagrams in a fraction of the time! Works seamlessly with Word® and PowerPoint® on PC or Mac.

- Natural, intuitive way to create math drawings.
- Quickly create accurate, to-scale figures.
- Easily paste your finished figures into tests and worksheets or use them in PowerPoint® presentations.
- With a single click, add leader lines and dimension arrows to specify lengths, angles, distances and radii.
- Requires PC running Windows® XP, Vista®, Windows® 7, or Intel based Macintosh®.

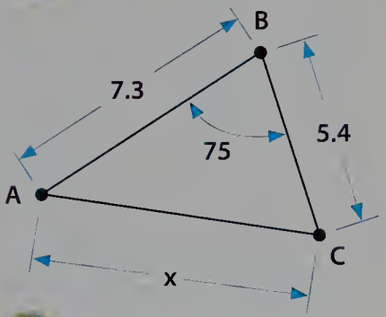
1 Sketch the figure



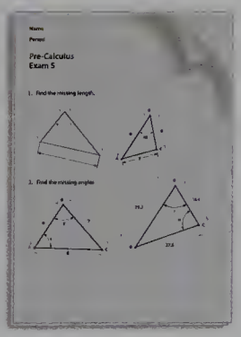
2 Add constraints



3 Create annotations

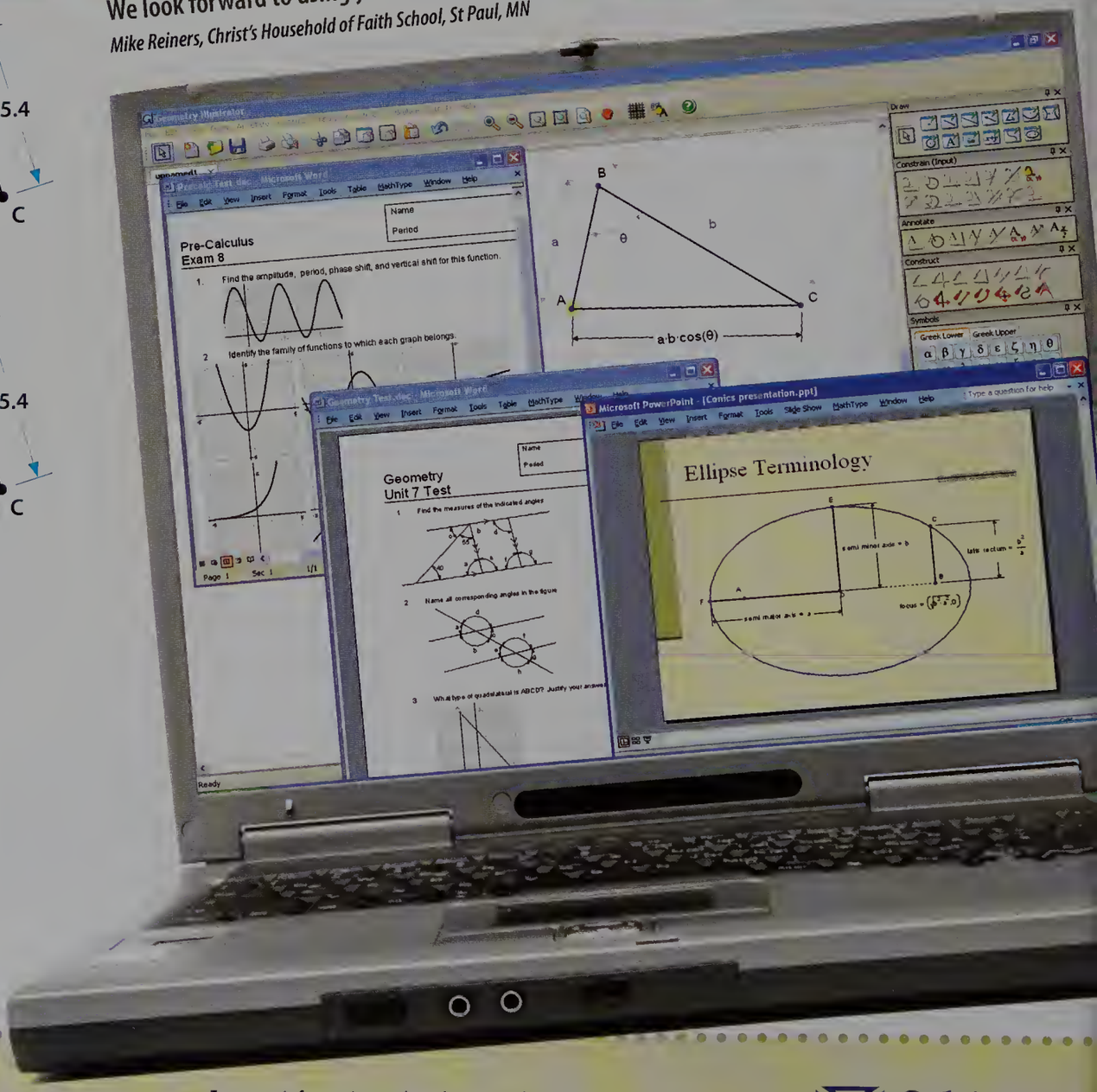


4 Copy and paste



"We gave Math Illustrations to the Minnesota State High School Mathematics League problem writing team to use on problems for the regular season and the state tournament. The program also worked very well for drawing the figures used during the Math Bowl at the tournament. We look forward to using your software for many years to come."

Mike Reiners, Christ's Household of Faith School, St Paul, MN



MATH ILLUSTRATIONS

For more information and to place an order, go to
www.mathillustrations.com

Saltire Software

contents

Volume 106, Number 8, April 2013

features

580 **Galileo, Gauss, and the Green Monster**
Dan Kalman and Daniel J. Teague
Using ideas of Galileo and Gauss but avoiding calculus, students create a model that predicts whether a fly ball will clear the famous left-field wall at Fenway Park.

586 **Geometric Reasoning about a Circle Problem**
Gloriana González and Anna F. DeJarnette
An open-ended problem about a circle illustrates how problem-based instruction can enable students to develop reasoning and sense-making skills.

592 **Building Sinusoids**
Mara G. Landers
A measurement-based activity can help students struggling to understand trigonometric functions.

608 **Students as Mathematics Consultants**
Jennifer L. Jensen
Five problems—relating to gas mileage, the national debt, store sales, shipping costs, and fish population—require students to use functions to connect mathematics to the real world.

614 **Is That Square Really a Circle?**
Christopher E. Smith
Considering circles in taxicab geometry helps students with Euclidean concepts.

620 **Elementary Algebra Connections to Precalculus**
Roberto López-Boada and Sandra Argüelles Daire
Students use elementary algebra concepts to solve trigonometric and logarithmic equations and systems.

on the cover

As the school year winds down and the baseball season winds up, two Dans—Kalman and Teague—show teachers of precalculus and second-year algebra how to bring ideas from calculus into their classes (pp. 580–85). Using ideas from Galileo and Gauss, discover what it will take for a batted baseball to clear Fenway Park's legendary Green Monster and involve your students in classical but relevant mathematics.

COVER PHOTO: ROB TRINGALI/GETTY IMAGES

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

EDITORIAL PANEL

SARAH SCHUHL, Centennial High School, Gresham, Oregon; *Chair*
DANE R. CAMP, 'Iolani School, Honolulu, Hawaii; *Board of Directors Liaison*
DAN CANADA, Eastern Washington University, Cheney
DAVID CUSTER, Decatur City High School, Decatur, Georgia
GREGORY D. FOLEY, Ohio University, Athens, Ohio
ALLISON GOLDEN, Champlain Valley Union High School, Hinesburg, Vermont
STAN IZEN, Latin School of Chicago, Chicago (*retired*)
SANDRA R. MADDEN, University of Massachusetts–Amherst
LAURIE H. RUBEL, Brooklyn College, CUNY, New York
GREG STEPHENS, Hastings High School, Hastings-on-Hudson, New York
ALBERT GOETZ, NCTM; *Staff Liaison*

JOURNALS STAFF

KEN KREHBIEL, *Associate Executive Director for Communications*
JOANNE HODGES, *Director of Publications*
ALBERT GOETZ, *Journal Editor*
GRETCHEN SMITH MUI, *Editor*
LUANNE M. FLOM, PAMELA A. GRAINGER, ELIZABETH M. SKIPPER, SARA-LYNN GOPALKRISHNA, *Contributing Editors*
SHEILA J. BARKER, *Review Services Assistant*
CHRISTINE A. NODDIN, *Publications Assistant*

MARKETING STAFF

JENNIFER J. JOHNSON, *Senior Director, Member Services, Marketing, and Business Development*
TOM PEARSON, *Sales Manager*

NCTM Board of Directors

LINDA M. GOJAK, John Carroll University, University Heights, Ohio; *President*
J. MICHAEL SHAUGHNESSY, Portland State University, Oregon; *Past President*
KICHOON YANG, NCTM; *Executive Director*
ROBERT Q. BERRY III, University of Virginia, Charlottesville
MARGARET (PEG) CAGLE, Lawrence Gifted Magnet School, Los Angeles Unified School District
DANE R. CAMP, 'Iolani School, Honolulu, Hawaii
ANNE M. COLLINS, Lesley University, Cambridge, Massachusetts
DEBBIE DUVALL, Elk Island Public Schools, Sherwood Park, Alberta
MARK W. ELLIS, California State University, Fullerton
KAREN J. GRAHAM, University of New Hampshire, Durham
GLADIS KERSAINT, University of South Florida, Tampa
LATRENDA KNIGHTEN, Polk Elementary School, Baton Rouge, Louisiana
MATTHEW R. LARSON, Lincoln Public Schools, Nebraska
JONATHAN (JON) WRAY, Howard County Public Schools, Maryland

To contact a journal staff person, e-mail **mt@nctm.org**.
For NCTM advertising sales, contact **The Townsend Group,** Kim Kelemen, national sales manager; Kkelemen@townsend-group.com; (301) 215-6710, ex. 103

Mission Statement: The National Council of Teachers of Mathematics is a public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

All correspondence should be addressed to the *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To send submissions, access **mt.msubmit.net**. Send letters to the editor to **mt@nctm.org**.

Permission to photocopy material from *Mathematics Teacher* is granted to persons who wish to distribute items individually (not in combination with other articles or works), for educational purposes, in limited quantities, and free of charge or at cost; to librarians who wish to place a limited number of copies on reserve; to authors of scholarly papers; and to any party wishing to make one copy for personal use. Permission must be obtained to use journal material for course packets, commercial works, advertising, or professional development purposes. Uses of journal material beyond those outlined above may violate U.S. copyright law and must be brought to the attention of the National Council of Teachers of Mathematics. For a complete statement of NCTM's copyright policy, see the NCTM website, **www.nctm.org**.

For information on **article photocopies or back issues**, contact the Customer Service department in the Headquarters Office.

A cumulative index appears on the NCTM website at **www.nctm.org/mt/mt-indexes.htm**. The *Mathematics Teacher* is indexed in *Biography Index, Contents Pages in Education, Current Index to Journals in Education, Education Index, Exceptional Child Education Resources, Literature Analysis of Microcomputer Publications, Mathematical Reviews, Media Review Digest,* and *Zentralblat für Didaktik der Mathermatik*.

Information is available from the Headquarters Office or online at www.nctm.org/membership regarding the three **other official journals**, *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of NCTM's services. Dues for individual membership are $81 (U.S.), which includes $34 for each *Mathematics Teacher* subscription. Each additional school journal (*Mathematics Teaching in the Middle School* and *Teaching Children Mathematics*) subscription is $34. Each additional subscription to the *Journal for Research in Mathematics Education* is $61. Foreign subscribers add $18 (U.S.) postage for the first journal and $4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org. © 2013. The National Council of Teachers of Mathematics, Inc. Printed in the U.S.A.

departments

564 **Reader Reflections**

570 **Media Clips**
Lotto Prize.1 // Lotto Prize.2
Ron Lancaster
Edited by Louis Lim and Lionel Garrison

574 **Mathematical Lens**
Archetypal Mathematics
Günhan Caglayan
Edited by Ron Lancaster and Brigitte Bentele

600 **May 2013 Calendar and Solutions**
Edited by Margaret Coffey and Art Kalish

624 **Technology Tips**
Podcasting Potential for High School Mathematics
Brandon R. Hanson
Edited by Larry Ottman and James Kett

630 **Delving Deeper**
A Closer Look at Function Transformations
Christopher V. Cappiello
Edited by Maurice Burke, J. Kevin Colligan, Maria Fung, and Jeffrey J. Wanko

637 **For Your Information**
Publications

640 **The Back Page: My Favorite Lesson**
Introducing Factorials
Marshall B. Lassak
Edited by Jennifer Wexler

calls for manuscripts

579 **2014 Focus Issue: What Makes Ideas Stick?**

599 **Assessment**

607 **Call for Chapters: *Using Research to Improve Instruction***

635 **Reasoning and Sense Making**

636 **Author Guidelines for *Mathematics Teacher***

624

more4U

Look online for these additional items:

- "Building Sinusoids" (p. 592): Examples of worksheets and student solutions
- Mathematical Lens (p. 574): An additional photograph for your own mathematical questions

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

in the next issue

Coming in May 2013:

- "Prepare for More Realistic Test Results," by Matthew R. Larson and Steven Leinwand
- "Graphing Polar Curves," by Jonathan F. Lawes
- "Exploring Logarithms with a Ruler," by Elliott Ostler
- NCTM Journals Survey Results

MINIMUM TIME, TWO REGIONS

My reflection "Generalizing a Problem: The Minimum Time between Two Points" (*MT* Dec. 2011/Jan. 2012, vol. 105, no. 5, p. 324) uses the following, hardly original, result. We can travel along a straight line path at velocity v. We start at point A, a units from the line, in a region where we can travel at velocity kv, with $0 < k < 1$. If we wish to travel to point B, which is c units "downline" from the foot of the perpendicular to the line, then, with $\sin\theta = k$ for $c \geq a\tan\theta$, the path that minimizes the travel time makes angle θ with the normal until it reaches the line at a point $a\tan\theta$ downline and then continues the rest of the distance, $c - a\tan\theta$, along the line.

Here we attach a second region in which we can travel at velocity lv, with $0 < l < 1$, on the side opposite the first. We wish to minimize the time required to reach point B, which is b units from the line in this second region, c units downline from our aforementioned point A (see **fig. 1 [Siegel]**).

From the previous reflection, it should be obvious that the correct path is to travel along the path that makes angle θ with the normal until we reach the boundary line. Next we travel distance $c - a\tan\theta - b\tan\varphi$, assuming that this quantity is nonnegative and that φ is defined by $\sin\varphi = l$, along the boundary line. Then we proceed straight to point B.

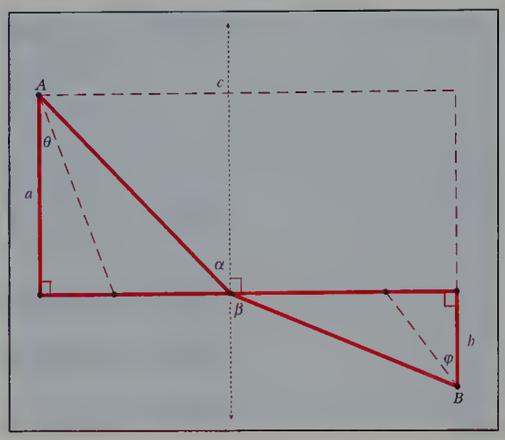


Fig. 1 (Siegel)

We deal with the case in which $c < a\tan\theta + b\tan\varphi$ a little differently. Here the journey has no "middle" portion. We travel to a point at angle α to the normal from point A until we reach the boundary, and then we travel at angle β to the normal until we reach point B. Our task is to find α and β. To that end, we immediately have one equation relating them: $a\tan\alpha + b\tan\beta = c$. To find another, we need to minimize $a\sec\alpha/kv + b\sec\beta/lv$ subject to that constraint. We set the derivative of

$$T = \frac{a\sec\alpha}{kv} + \frac{\sqrt{b^2 + \left(c - a\tan\alpha\right)^2}}{lv}$$

equal to 0 and obtain

$$\frac{\tan\alpha}{k} = \frac{\sec\alpha\left(c - a\tan\alpha\right)}{l\sqrt{b^2 + \left(c - a\tan\alpha\right)^2}} = \frac{\sec\alpha\sin\beta}{l},$$

which leads to Snell's law:

$$\frac{\sin\alpha}{k} = \frac{\sin\beta}{l}$$

Using our two equations to find α and β is not difficult numerically if we have specific values for a, b, c, k, and l; in general, doing so is prohibitive. Finally, note that for any nonnegative value of c, we obtain the same equations if it is understood at the outset that we are not allowed to travel along the boundary because of refraction.

Steven Siegel
siegel443@comcast.net
Niagara University (retired)
Niagara, NY, Feb. 13, 2012

AND ONE TO GROW ON: UNWRAPPING THE BIRTHDAY PROBLEM

Kudos to Alfinio Flores and Kevin M. Cauto for their use of simulation to explore the birthday problem ("Empirical Approaches to the Birthday Problem," *MT* Sept. 2012, vol. 106, no. 3, pp. 132–37). Their concluding comment that 23 birthdays spread evenly throughout the year translates to 2 per month can be used further to provide a "lower bound" to the probability of at least one match.

Two people born in the same month have a match probability of 1/30. (For a simple approximation, we treat all months as having 30 days.) Thus, the expected value of the number of matches for all 12 months is 12/30, which is .4, which is "surprisingly" near .5. Students know that real-life birthdays may not be perfectly evenly spaced and can explore the idea that the probability of at least one match only gets higher as the birthdays have more clustering. For example, if we move one month's 2 birthdays into another month (so that we have 10 months with 2 birthdays, one with 4, and one with 0), the expected number of matches for the year becomes (10/30) + (1 – (29/30) · (28/30) · (27/30)) ≈ .52.

The tool of recursion that the authors use to obtain the theoretical probability from a calculator program can also yield a spreadsheet exploration (as described in my article "Exploring the Birthday Problem with Spreadsheets" [*MT* May 1999, vol. 92, no. 5, pp. 407–11]). That latter article also provides further insight into the answer of 23 by using the modeling

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

approximation of "number of opportunities" (i.e., the number of pairs of people in the room that could yield a match).

Lawrence M. Lesser
Lesser@utep.edu
The University of Texas at El Paso
Aug. 23, 2012

GLOBAL MATHEMATICS

In retirement, I have flashbacks to some of my favorite class activities over the years. Some current teachers might be interested in one project that younger students enjoyed.

My goal was to illustrate the commonality of mathematics ideas around the world—in particular, notation. Each of my first-year algebra students researched an address of an education-related department (or the address of the U.S. embassy) for each of five countries of their choosing. Then they composed letters introducing themselves (age, grade level, and course), why they were writing (the project's goal), and what they were requesting (a copy of a textbook used by students of similar demographics in that country).

My class received a number of textbooks from many different countries. Students were amazed at how many examples and problems were familiar to them, even though the words were not in English—and the exercise in practical English composition was a bonus.

Margaret Cibes
cibesm@comcast.net
University of Hartford (retired)
Hartford, CT, May 19, 2012

PABLO'S THEOREM

On a recent test in my problem-solving class, I gave the following problem:

> The number 423A05BA is divisible by 99. Find all possible values of A and B where A and B are single digits from 0 to 9.

I had expected that students would realize that if a number were divisible by 99, then it had to be divisible by both 9 and 11. If a number is divisible by 9, then the sum of its digits is divisible by 9, or $14 + 2A + B$ is divisible by 9, and, hence, $14 + 2A + B = 18$, 27, or 36. This yields

$$2A + B = 4, 13, \text{ or } 22. \qquad (1)$$

If the original number is divisible by 11, then the sum of the digits in the odd-numbered positions minus the sum of the numbers in the even-numbered positions is divisible by 11. The result, $B - 2A$, is divisible by 11 and, hence,

$$B - 2A = -11 \text{ or } 0. \qquad (2)$$

Adding (1) and (2), we find that $2B$ can be –7, 4, 2, 13, 11, or 22. Only $2B = 4$ and $2B = 2$ work. So there are two solutions: $A = 1$ and $B = 2$, or $A = 6$ and $B = 1$.

One of my students, Pablo Garcia, made up his own rule. He said that if a number is divisible by 99, then if you pair up the digits, the sum has to be 99. So

$$42 + 3A + 05 + BA = 99. \qquad (3)$$

Looking at the units digit of the sum in (3), we see that $7 + 2A = 9$ or 19. When $7 + 2A = 9$, $A = 1$; looking at the tens digit, we see that $7 + B = 9$, implying that $B = 2$. Considering the case when the sum of the units digit in (3) is 19, we obtain $7 + 2A = 19$, so $A = 6$; looking at the units digit (and realizing that we have to carry a 1), we find that $1 + 4 + 3 + 0 + B = 9$, so $B = 1$.

When I suggested that 9999, which is clearly divisible by 99 but whose digits sum to 198, disproves his rule, Pablo modified it: When paired, the sum of the digits has to be a *multiple* of 99. He continued: The sum in (3) in this case couldn't be, say, 198, since the first number in (3) is 42, the maximum value of the second number is 39, the third number is 5, and the maximum value of the fourth number is 99. So the maximum value of the sum in (3) is 185. Therefore, the sum (3) can't be any multiple of 99 other than 99.

I asked, "How do you know that the sum of the digits when paired is a multiple of 99?" Pablo answered, "I checked a ton of cases, and that was always true." He then sent me a list of the first 10,000 multiples of 99 that he had obtained from the Internet. All multiples of 99 with an even number of digits did have the property that the sum of the digits in pairs was a multiple of 99.

But, as I pointed out, the 111th number on his list, 10989, did not have that property that he stated. Pairing the digits (as much as possible) and adding the results, we get $10 + 98 + 9$, which is not a multiple of 99. But Pablo said that I had not applied his rule correctly: I had to turn it into a number with an even number of digits by putting a 0 in front. So the sum of the pairs is now $01 + 09 + 89$, which *is* a multiple of 99.

This was starting to get interesting. Perhaps Pablo had happened upon a valid test. I set out to prove it.

We begin with a number, N, having an even number of digits. (If it has an odd number of digits, add a 0 to the front of it to make it a number with an even number of digits.) Now suppose that the number can be broken into n pairs; let's call the rightmost pair p_1, the next pair to the left p_2, the next pair to the left p_3, and so on until we get to the last pair, p_n. (So if we have the number 535590, $p_1 = 90$, $p_2 = 55$, and $p_3 = 53$.) Then the number N can be written as $N = p_1 + 100p_2 + 10000p_3 + \cdots + 10^{2n-2}\, p_n$ where each coefficient is an even power of 10.

Recall that a number is divisible by 99 if it is congruent to zero mod 99. So we will try to determine when N is congruent to zero mod 99. Realizing that an even power of 10 is congruent to 1 mod 99, we find that $N \equiv p_1 + 100p_2 + 10000p_3 + \cdots + 10^{2n-2}\, p_n \pmod{99} \equiv p_1 + p_2 + p_3 + \cdots + p_n \pmod{99}$.

So if N is divisible by 99 (that is, if N is congruent to zero mod 99), then $p_1 + p_2 + p_3 + \cdots + p_n$ is also congruent to zero mod 99, meaning that the sum of the numbers formed by pairing the digits is divisible by 99. Conversely, if the sum of the numbers formed by pairing the digits is divisible by 99, then so is N.

So we have Pablo's rule: A number is divisible by 99 if and only if the sum of the numbers formed when the digits are paired is divisible by 99. (If the number has an odd number of digits, put a zero before it when applying the rule.) Thanks to Pablo for these stimulating insights and for teaching his teacher!

Alan Sultan
asultan956@aol.com
CUNY–Queens College
Flushing, NY, May 28, 2012

COMPLETING THE SQUARE

I have found the following approach helpful to students who struggle with the concept of completing the square. Suppose that we start with $x^2 + 6x$ and wish to find the constant that will complete the square.

We use a diagram that the students used when they first started factoring, and with the information given they can complete much of the diagram (see **fig. 1 [Metz]**). Since both factors are the same, $6x$ comes from adding $3x$ and $3x$ (see **fig. 2 [Metz]**). Then it is an easy task to complete the factors (see **fig. 3 [Metz]**).

It readily appears that the constant is 9—that is, $(x + 3)(x + 3) = x^2 + 6x + 9 = (x + 3).^2$ Students can see that they took half of 6 to get 3 and that 3 was multiplied by 3, producing 9.

James Metz
metz@hawaii.edu
Kapi'olani Community College
Honolulu, HI, May 3, 2012

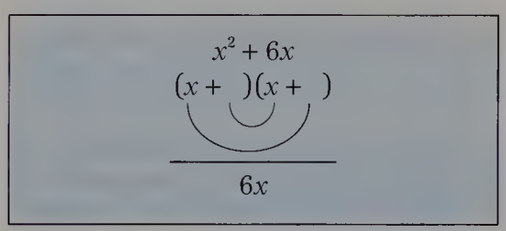


Fig. 1 (Metz)

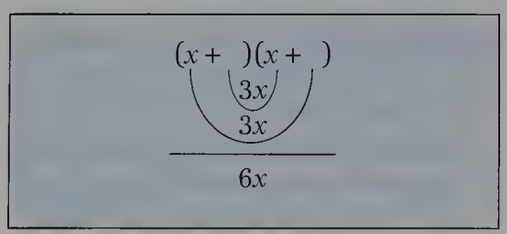


Fig. 2 (Metz)

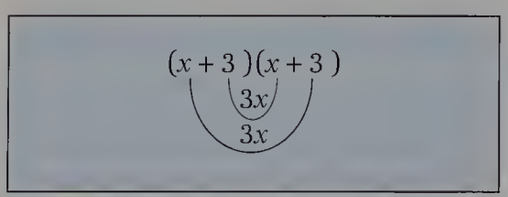


Fig. 3 (Metz)

JOE DIMAGGIO AND THE "AT LEAST ONCE" RULE OF PROBABILITY

Like many teachers, I try hard to find real-world examples of mathematical principles, examples that today's students can relate to. For a long time, I had difficulty finding a suitable example of the "at least once" rule of probability. I just couldn't see my students drawing at least one ace from a deck of cards in ten tries. I finally developed the following example, which is now one of my favorite lessons.

In 1941, Joe DiMaggio, the New York Yankees Hall of Fame center fielder, broke the record for hitting safely in consecutive games. DiMaggio broke Wee Willie Keeler's record of 44 games, set way back in 1897. What was the probability of DiMaggio getting a base hit in 56 consecutive baseball games?

To solve this problem, we first need to determine the probability that DiMaggio will get at least one hit in a baseball game. DiMaggio ended the 1941 season with a batting average of .357, meaning that, on average, each time that he was at bat he had a 35.7% chance of getting a base hit. Clearly, then, the probability of *not* getting a hit was 1 – .357, or .643. In other words, for each time that he was at bat, DiMaggio had a 64.3% chance of not getting a hit.

DiMaggio had 541 official at-bats during 139 games that season, or just under 4 at-bats per game. If we make the simplifying assumption of exactly 4 at-bats in every game, we need to determine the probability of someone with a .357 batting average not getting at least one hit in 4 tries. That is $P(\text{none in 4}) = .643 \cdot .643 \cdot .643 \cdot .643 = .643^4$, and the probability of getting at least 1 hit in 4 at-bats is 1 minus that, or $P(\text{at least 1 in 4}) = 1 - (.643)^4 \approx .829$.

Therefore, for a batter with 4 at-bats per game and a batting average of .357, the probability of getting at least 1 hit in 56 consecutive games is $P(\text{at least 1 in 56 games}) = (.829)^{56} \approx .000027$, or 27 chances per million.

DiMaggio did get a hit in his 56th consecutive game, setting the record that still stands today, seventy-one years later. (The closest anyone has come to it since then was Pete Rose, who hit in 44 consecutive games in 1978.)

Interestingly, DiMaggio still had nearly an 83% chance of getting a hit in his next game as well but failed to do so. Notice that it does not matter whether he had a hit in 56 consecutive games; the probability of getting a hit in the 57th was still the same. However, the law of large numbers finally caught up with him.

Bob Pitlak
Robert.Pitlak@Castleton.edu
Castleton State College
Castleton, VT, June 13, 2012

PROBLEMS WITH PEMDAS

During my four decades of teaching second-year algebra, I have encountered a trio of problems with the widely taught mnemonic device known as PEMDAS. Now my routine is to correct those problems in the first lesson of each school year.

First, I present the problem of simplifying $(3)(2) \div (3)(2)$ and ask students to hold up the correct number of fingers to indicate the answer. Invariably, most if not all hold up exactly one finger each. Their thinking is that $(3)(2) \div (3)(2) = 6/6 = 1$. Following the PEMDAS-directed order of operations, they have erroneously eliminated the parentheses first. After students have entered the problem into direct-entry calculators so that it is displayed exactly as it was presented, they are surprised that the answer shown is 4. Finally, in group discussion, we analyze the discrepancy.

Of course, the error stems from the fact that in order of operations, parentheses mean grouping or multiplication (or both). If the problem had been presented as $(2 + 1)(2) \div (3)(2)$, then the first set of parentheses would have meant both grouping and multiplication.

However, there are no grouping symbols in $(3)(2) \div (3)(2)$. The multiplication and division indicated are properly executed in left-to-right order, so that $(3)(2) \div (3)(2) = 6/(3)(2) = 2(2) = 4$. We see then that the first problem with PEMDAS is that the P stands for *parentheses*. My students, instead, learn to use FGSEMDAS—*formed grouping symbols*, *exponents*, *multiplication*, *division*, *addition*, and *subtraction*.

In the second part of this lesson, I ask students to simplify

$$\frac{2+4}{12},$$

in which the operations indicated are addition and division. The second problem with PEMDAS is that it is silent on how properly to execute the addition

in this problem ahead of the division. FGSEMDAS comes to the rescue. The horizontal bar, called a *vinculum*, is also a grouping symbol. My students learn to express parentheses as the formed grouping symbols that the vinculum means; then they proceed as FGSEMDAS directs:

$$\frac{2+4}{12} = \frac{(2+4)}{12} = \frac{6}{12} = \frac{1}{2}$$

Also in this first lesson, I ask students to hold up the correct number of fingers to indicate the answer to this problem:

$$\sqrt{9+16}$$

Invariably, at least one student holds up seven fingers. Again, in group discussion we analyze the error. Thinking of the radical as an implied rational exponent, the student has used PEMDAS erroneously to execute the exponent, or radical, ahead of the addition. This error illustrates yet a third problem with PEMDAS.

Once again, FGSEMDAS is the hero. My students learn to initiate it by expressing parentheses as the formed grouping symbols that the vinculum means. Then they proceed according to FGSEMDAS:

$$\sqrt{9+16} = \sqrt{(9+16)} = \sqrt{25} = 5$$

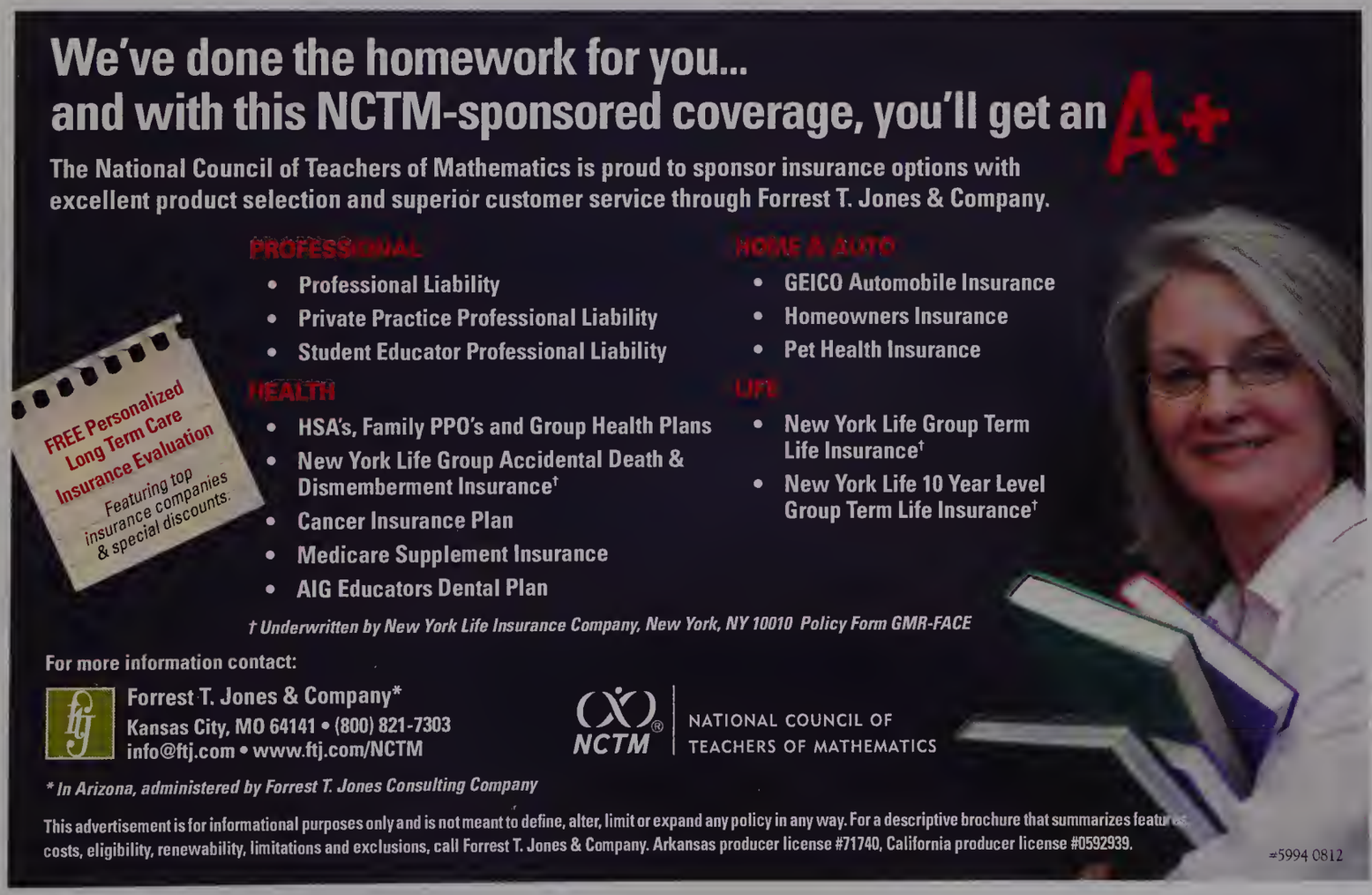
Clearly, FGSEMDAS offers a substantial advantage over PEMDAS. Although other students may follow *Please Excuse My Dear Aunt Sally,* my students and I use *For Goodness' Sake, Excuse My Dear Aunt Sally.*

James R. Rogers
james.rogers@mcschools.net
Neville High School
Monroe, LA, Sept. 1, 2012

WHERE ON THE GRAPH ARE THE COMPLEX ROOTS?

When students solve quadratic equations, they usually go through the process of using the quadratic formula as well as finding the roots using a graphing calculator. At my school, many teachers also use The Geometer's Sketchpad® (GSP) to project these graphs onto a whiteboard. Students can easily see that the numerical values of the roots are the x-intercepts of these graphs. They also have no trouble "seeing" that each x-intercept involves the identical amount added to or subtracted from the x-value of the vertex of the parabola. But they often wonder how to "see" the complex roots of a parabola that does not cross the x-axis. Does this case also involve symmetry around the vertex?

A few years ago, Doug Smith, who teaches mathematics at Arthur P. Schalick High School in Pittsgrove, New Jersey, provided a way to locate on the coordinate plane the complex roots of parabolas that do not intersect the x-axis. Using GSP, Smith graphed a typical parabola with complex roots—specifically, $y = x^2 - 4x + 7$ (see **fig. 1 [Schuloff Bonanomi]**). He then reflected this parabola over the horizontal line tangent to its vertex. This graph intersects the x-axis twice. Smith then constructed the circle whose diameter


We've done the homework for you...
and with this NCTM-sponsored coverage, you'll get an A+
The National Council of Teachers of Mathematics is proud to sponsor insurance options with excellent product selection and superior customer service through Forrest T. Jones & Company.
PROFESSIONAL
• Professional Liability
• Private Practice Professional Liability
• Student Educator Professional Liability
HOME & AUTO
• GEICO Automobile Insurance
• Homeowners Insurance
• Pet Health Insurance
HEALTH
• HSA's, Family PPO's and Group Health Plans
• New York Life Group Accidental Death & Dismemberment Insurance†
• Cancer Insurance Plan
• Medicare Supplement Insurance
• AIG Educators Dental Plan
LIFE
• New York Life Group Term Life Insurance†
• New York Life 10 Year Level Group Term Life Insurance†
FREE Personalized Long Term Care Insurance Evaluation
Featuring top insurance companies & special discounts.
† Underwritten by New York Life Insurance Company, New York, NY 10010 Policy Form GMR-FACE
For more information contact:
Forrest T. Jones & Company*
Kansas City, MO 64141 • (800) 821-7303
info@ftj.com • www.ftj.com/NCTM
NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
* In Arizona, administered by Forrest T. Jones Consulting Company
This advertisement is for informational purposes only and is not meant to define, alter, limit or expand any policy in any way. For a descriptive brochure that summarizes features, costs, eligibility, renewability, limitations and exclusions, call Forrest T. Jones & Company. Arkansas producer license #71740, California producer license #0592939.
#5994 0812

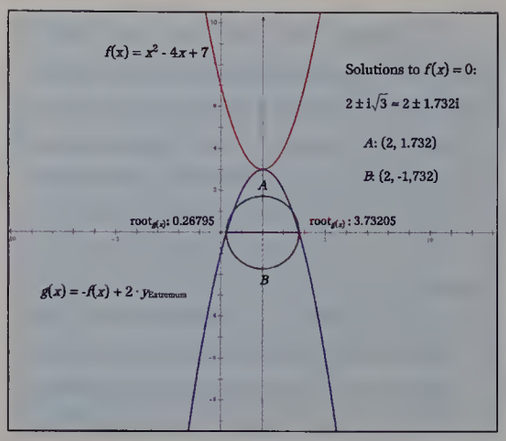


Fig. 1 (Schuloff Bonanomi)

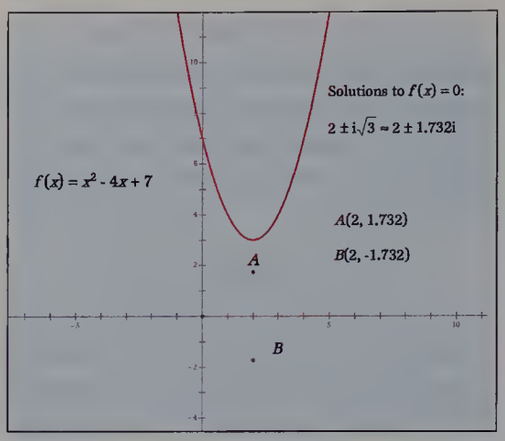


Fig. 2 (Schuloff Bonanomi)

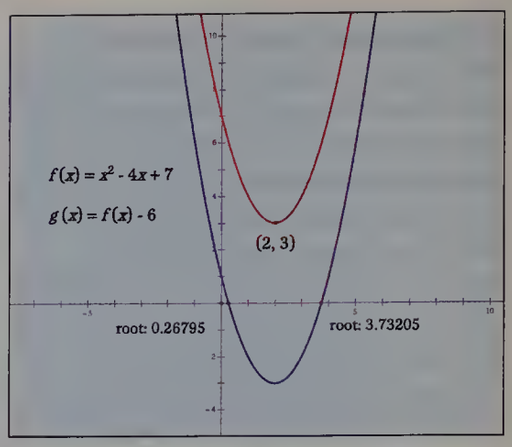


Fig. 3 (Schuloff Bonanomi)

is the line segment connecting the two roots of the reflected parabola. The coordinates of the two intersections of this circle with a vertical diameter are numerically equal to the components of the roots of the original parabola. These components can also be seen in **figure 1 (Schuloff Bonanomi)**. Thus, we can locate on the coordinate plane the complex roots of the parabola (see **fig. 2 [Schuloff Bonanomi]**).

To verify that these coordinates actually are the roots, I changed the graph somewhat to make the algebraic steps of the proof easier to follow.

We are given $y = ax^2 + bx + c$ and roots of

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Since $b^2 - 4ac$ will be negative, the actual roots will be

$$x = -\frac{b}{2a} \pm \frac{i\sqrt{4ac - b^2}}{2a}.$$

We need to show that the expression $\sqrt{4ac - b^2}/(2a)$, which is the y-component of the roots $x \pm yi$, is the length of the radius of the circle.

First, to write the equation of the reflected parabola, we can write the equation of its negation (because both will have the same root). The only difference between the equation of the original parabola and that of the new, translated one will be the c-value. The c-value (y-intercept) of the new parabola will be twice the y-value of the vertex of the original parabola subtracted from c (y-intercept of the original parabola [see **fig. 3 [Schuloff Bonanomi]**).

We start by finding the y-value of the vertex:

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= a\left(\frac{b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} \\ &= \frac{4ac - b^2}{4a} \end{aligned}$$

(This is the y-value of the vertex of the original parabola.)

The equation for our new parabola is then

$$\begin{aligned} y &= ax^2 + bx + c - 2\left(\frac{4ac - b^2}{4a}\right) \\ &= ax^2 + bx + c - \frac{8ac - 2b^2}{4a} \\ &= ax^2 + bx + \frac{4ac - 8ac + 2b^2}{4a} \\ &= ax^2 + bx + \frac{-4ac + 2b^2}{4a} \\ &= ax^2 + bx + \frac{b^2 - 2ac}{2a}. \end{aligned}$$

Set the quadratic expression above equal to zero and solve for x using the quadratic formula:

$$\begin{aligned} x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4a\left(\frac{b^2 - 2ac}{2a}\right)}}{2a} \\ &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 2\left(b^2 - 2ac\right)}}{2a} \\ &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 2b^2 + 4ac}}{2a} \\ &= \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} \end{aligned}$$

The length of the radius of the circle is half the distance between the two roots. We need only subtract one root from the other and divide by 2 (because the roots are lined up horizontally). Subtracting $-b/(2a)$ from itself will produce 0; doubling and then halving the radical expression leaves the result

$$\frac{\sqrt{4ac - b^2}}{2a}.$$

This expression, since it is the length of the radius of the circle, becomes the y-component of the complex roots of the original parabola.

Anita Schuloff
aschuloff@paramus-catholic.org
Mark Bonanomi
mbonanomi@paramus-catholic.org
Paramus Catholic High School
Paramus, NJ, July 24, 2012

DO YOU HAVE SOMETHING TO ADD?

Share with readers and the Editorial Panel your opinions about any of the articles or departments appearing in this issue by writing to Reader Reflections. Letters should be sent to mt@nctm.org.

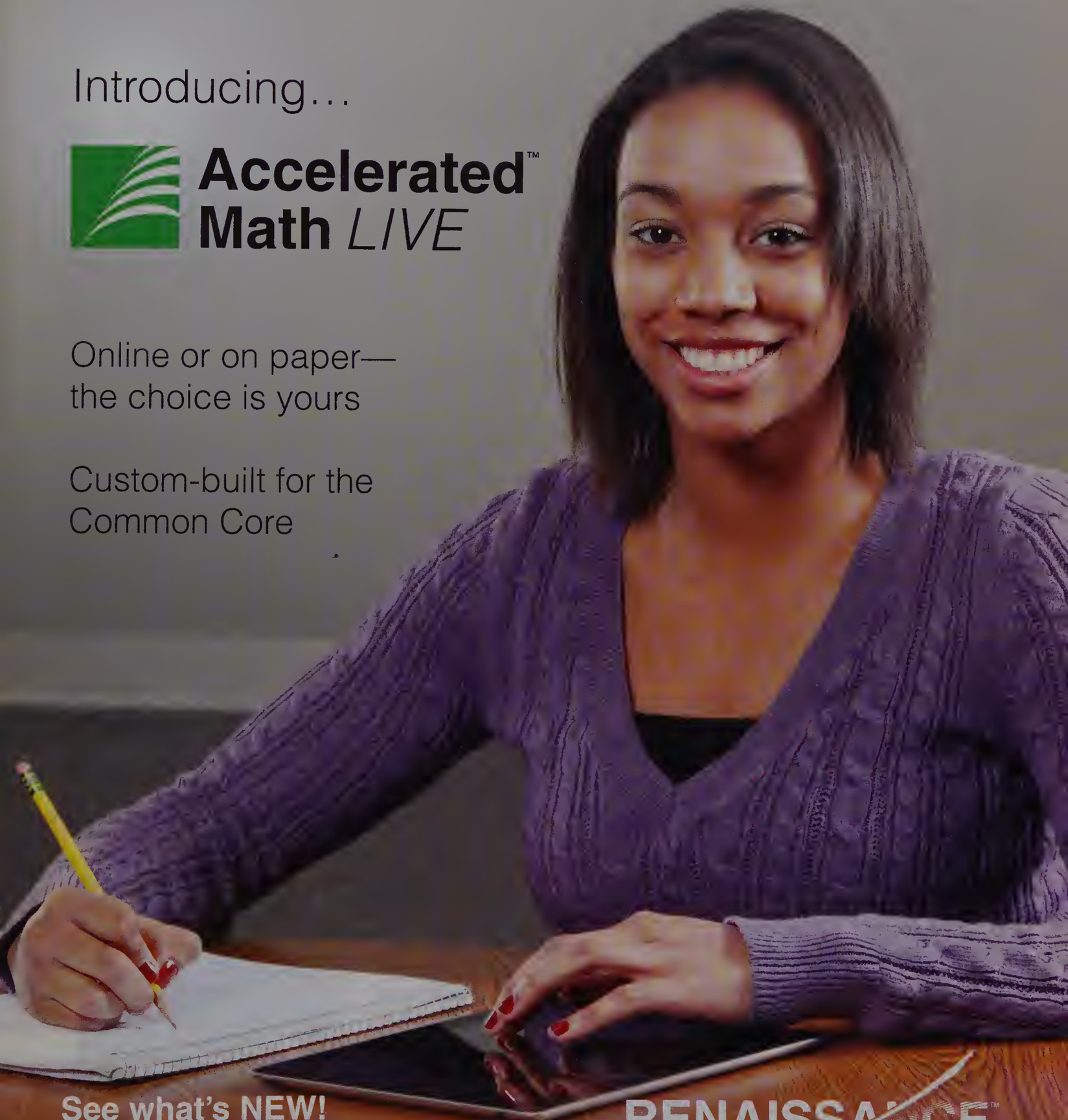
Bring personalized practice to life!

Introducing...

Accelerated™ Math *LIVE*

Online or on paper—
the choice is yours

Custom-built for the
Common Core



See what's NEW!
www.renlearn.com/lp/21252
(800) 338-4204, ref. #21252

RENAISSANCE™
LEARNING
Accelerating learning for all

Ron Lancaster

Lotto Prize.1

PAGADESIGN/ISTOCKPHOTO

John Kinzel of Paradise Hill, Sask., used 4s when picking numbers for the $40 Million Lotto MAX. He chose 4, 14, 24, 41, 43, 46, and 49. If you add up the separate digits, [they add] up to 40, which then adds up to a 4.

4 + 1 + 4 + 2 + 4 + 4 + 1 + 4 + 3 + 4 + 6 + 4 + 9 = 40
4 + 0 = 4

The date of the $40 Million Lotto Max draw was August 19, 2011. The date in numbers only was 8/19/2011.

8 + 1 + 9 + 2 + 0 + 1 + 1 = 22
2 + 2 = 4

Was there something supernatural about John Kinzel focusing on 4s? Who cares—he and his wife are $40 Million RICHER!

Source: "John Kinzel of Paradise Hill Was Ready "FOUR" the $40 Million Lotto MAX Win," by Lucky Lottery Squirrel, August 28, 2011, http://lotterysquirrel.com/2011/08/28/john-kinzel-of-paradise-hill-was-ready-four-the-40-million-lotto-max-win/

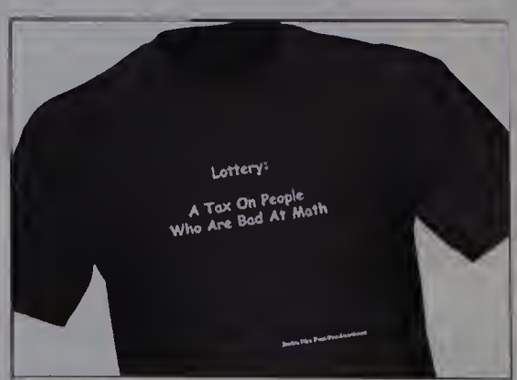
Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

Edited by ***Louis Lim****, louis.lim1@gmail.com*
Thornhill Secondary School, Thornhill, ON, Canada

Lionel Garrison*, garrison@horacemann.org*
Horace Mann School, Bronx, NY

1. Readers of this article may have noticed that Lottery Squirrel made an error in addition: The digits in the seven winning numbers sum to 50, not 40. But is it possible to find seven different numbers from 1 to 49, each containing at least one 4, the sum of whose digits do add up to 40?

2. Consider your answers to questions 1 and 2 from "Lotto Prize.2." Do you agree or disagree with the statement on the T-shirt? See **figure 1 ("Lotto Prize.1")**. Explain your answer.



Source: http://shop.cafepress.com/lotteries-bad-math

Fig. 1 ("Lotto Prize.1")

Lotto Prize.2

We've heard "lucky number 7," but how about lucky number 4?

John Kinzel [a resident of the oil and gas community of Paradise Hill, Saskatchewan] only picked numbers that included "four" when he bought a ticket for the $40 Million LOTTO MAX draw....

Every winning number had a four in it. John bought his ticket on the day of the draw and checked it with his wife on the Internet the next morning. When his wife, Darlene, started reading off the winning numbers—4, 14, 24, 41, 43, 46, and 49—John says he knew the result would be a good one.

"I knew those numbers sounded familiar," said John. "When I got the first three I thought OK. The rest of them are the same. Then I asked her how many winners there were, and she said, 'Just one.' Then I said, 'Heck, looks like we won $40 million.'"

The winning ticket was bought in Lloydminster.

Another Quick Pick Selection on the ticket also matched four of John's lucky "four" numbers to win an additional $20, bringing the couple's total winnings to $40,000,020....

Paradise Hill is a village of 486 people located 300 kilometres northwest of Saskatoon and 64 kilometres northeast of Lloydminster.

Source: "Couple Wins Biggest Lotto Prize in Sask History," Global News Regina, Friday, August 26, 2011, http://www.globalregina.com/couple+wins+biggest+lotto+prize+in+sask+history/6442470855/story.html

1. (*a*) To buy a ticket for the Lotto Max lottery, a player must select 7 different numbers from 1 to 49. How many ways are there of selecting these numbers?

   (*b*) If the 7 numbers that a player picks match the 7 numbers that are selected at random by the Lotto Max organization, then the player will win the grand prize of the lottery. What is the probability of winning the grand prize?

   (*c*) Determine how many ways there are of picking the 7 numbers if each number contains at least one digit that is a 4.

   (*d*) Determine how many ways there are of picking the 7 numbers if each number contains at least one digit of each of the following:

   (i) 1
   (ii) 2
   (iii) 3
   (iv) 5
   (v) 6
   (vi) 7
   (vii) 8
   (viii) 9

2. (*a*) Lotto 649 is another lottery in Canada that is identical to Lotto Max except that players pick 6 numbers from 1 to 49. What is the probability of winning the grand prize? Are the chances of winning the grand prize in the Lotto 649 better than, worse than, or the same as winning the grand prize in Lotto Max?

   (*b*) Suppose that somewhere in the world is a lottery called Lotto 4349 in which a player chooses 43 numbers from 1 to 49. Compare the chances of winning the grand prize for this lottery with the chances of winning the grand prize in Lotto 649.

3. (*a*) **Figure 1 ("Lotto Prize.2")** shows a map of the region that surrounds Paradise Hill, Lloydminster, and Saskatoon, the three municipalities mentioned in the article. Points *L* and *S* represent the locations of Lloydminster and Saskatoon, respectively. Use the information given in the article to locate Paradise Hill.

   (*b*) Suppose that another journalist who covered this story wanted to include information about the location of Paradise Hill in relation to Lloydminster and Saskatoon and did so as follows:

   "Paradise Hill is a village of 486 people located 243 kilometres from Saskatoon and 47 kilometres from Lloydminster."

   These distances are the exact distances between the municipalities ("as the crow flies"), not how far one would travel by car. Using just this information, can you determine the exact location of Paradise Hill?

Fig. 1 ("Lotto Prize.2")

MEDIA CLIPS solutions

"Lotto Prize.1" answers

1. Readers are invited to analyze the list of 3,432 cases from question 1(*c*) of "Lotto Prize.2" to find cases in which the sum of the digits is equal to 40. The first person to e-mail Ron Lancaster, ron2718@nas.net, with the correct results will receive a beautiful collection of 40 photos of the number 40.

2. Opinions will vary.

"Lotto Prize.2" answers

1. (*a*) The number of ways of choosing 7 different numbers from 1 to 49 is

$$\binom{49}{7},$$

which is equal to 85,900,584:

$$\binom{49}{7} = \frac{49!}{7!42!} = 85{,}900{,}584$$

(*b*) The probability of winning the grand prize is equal to

$$\frac{1}{85{,}900{,}584} \approx 1.16 \cdot 10^{-8},$$

or approximately .0000000116.

(*c*) Fourteen numbers contain at least one 4. The numbers are 4, 14, 24, 34, 40, 41, 42, 43, 44, 45, 46, 47, 48, and 49. The number of ways of picking 7 numbers from this group of 14 numbers is ${}_{14}C_7$, which is equal to 3,432:

$${}_{14}C_7 = \frac{14!}{7!(14-7)!} = 3{,}432$$

(*d*) There are 3,432 ways of picking 7 different numbers for which each number contains at least one digit that is a 3. The answer is the same for the case in which at least one digit is a 1 or a 2. It is impossible to pick 7 different numbers for which each number has at least one 5 or one 6 or one 7 or one 8 or one 9.

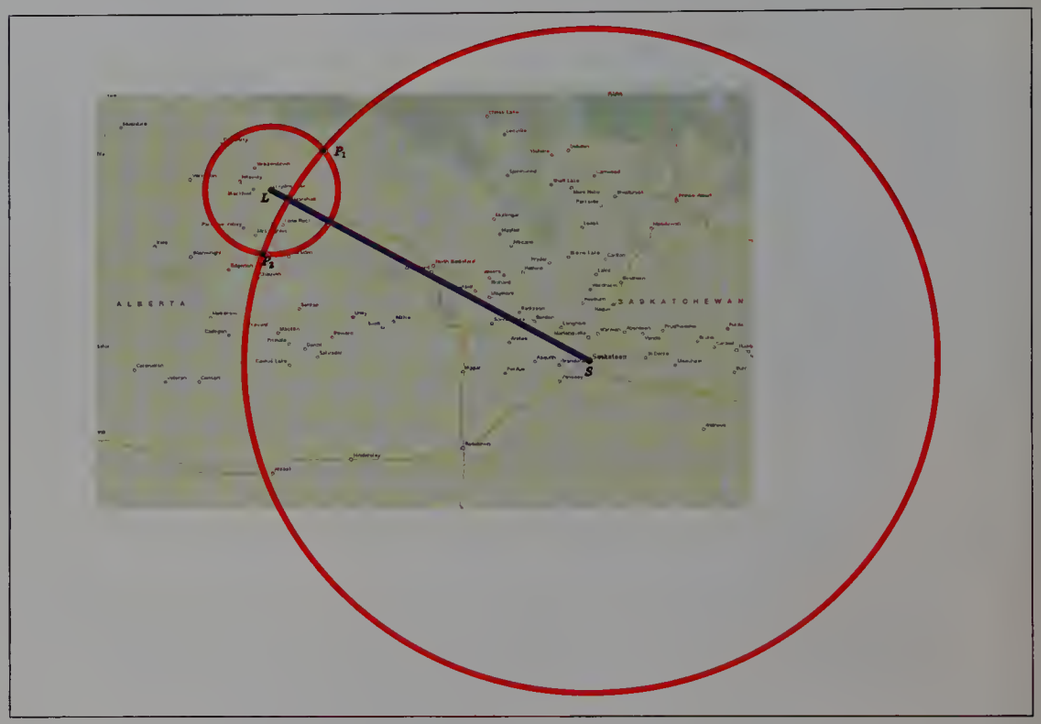


Fig. 2 ("Lotto Prize.2")

2. (*a*) The number of ways of choosing 6 different numbers from 1 to 49 is ${}_{49}C_6$, which is equal to

$$\frac{49!}{6!(49-43)!},$$

or 13,983,816. The probability of winning the grand prize is equal to

$$\frac{1}{13{,}983{,}816} \approx 7.15 \cdot 10^{-8},$$

or approximately .0000000715. The chances of winning are better for this lottery.

(*b*) The number of ways of choosing 43 different numbers from 1 to 49 is ${}_{49}C_{43}$, which is equal to

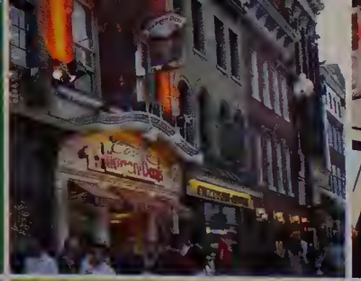
$$\frac{49!}{43!(49-6)!},$$

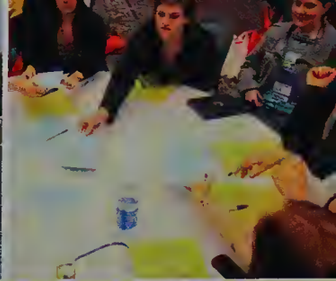
or 13,983,816. This is the same answer as in part (*a*).

3. (*a*) Without a scale, answers will vary. See part (*b*) for the exact location of Paradise Hill (point P_1).

(*b*) If this is all the information we are given, it is possible to narrow down the location of Paradise Hill to two locations. One way is to paste a map of the region into a Geometer's Sketchpad® file and use the measuring tools and ratios to construct two circles (see **fig. 2 ["Lotto Prize.2"]**). It is not possible to decide which point of intersection is the location of Paradise Hill; more information would be needed.

RON LANCASTER, ron2718@nas.net, is a senior lecturer at the Ontario Institute for Studies in Education of the University of Toronto. His interests include photographs and videos of mathematics, Math Trails, magic, and puns.



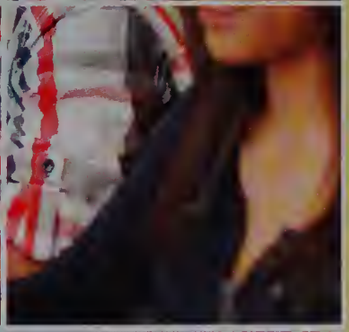


August 1–3, 2013 | Washington, D.C.

Engaging Students *in* Learning:
Mathematical Practices & Process Standards

AN NCTM INTERACTIVE INSTITUTE FOR GRADES 9–12

Register by MAY 31 and save $40!



Engage Your Students in Learning

You need the best strategies to prepare your students for success, and NCTM's Interactive Institute an High Schaal Mathematics offers a variety af activities and tactics to effectively address the Camman Core mathematical practices and NCTM Pracess Standards—giving yaur students better oppartunities to examine, interpret, and think critically abaut math cancepts.

The experience will be suited to yaur interests—you'll take part in sessions and be grauped with educatars according to yaur strand af facus:

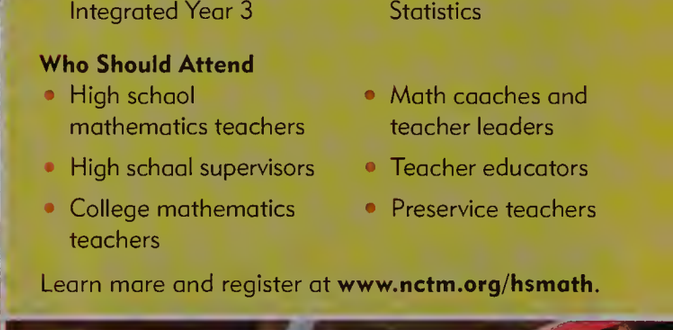
Strands

- Algebra 1/ Integrated Year 1
- Algebra 2/ Integrated Year 3
- Geametry/ Integrated Year 2
- Probability and Statistics

Who Should Attend

- High schaal mathematics teachers
- High schaal supervisors
- College mathematics teachers
- Math caaches and teacher leaders
- Teacher educators
- Preservice teachers

Learn mare and register at **www.nctm.org/hsmath.**



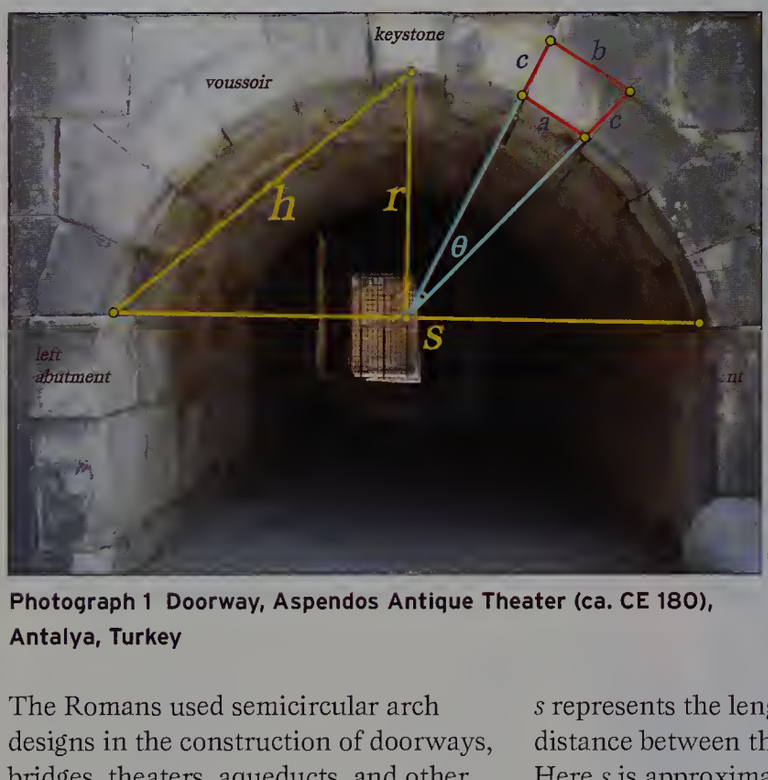
NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

(800) 235-7566 | WWW.NCTM.ORG

Space is limited—
REGISTER TODAY! **www.nctm.org/hsmath**

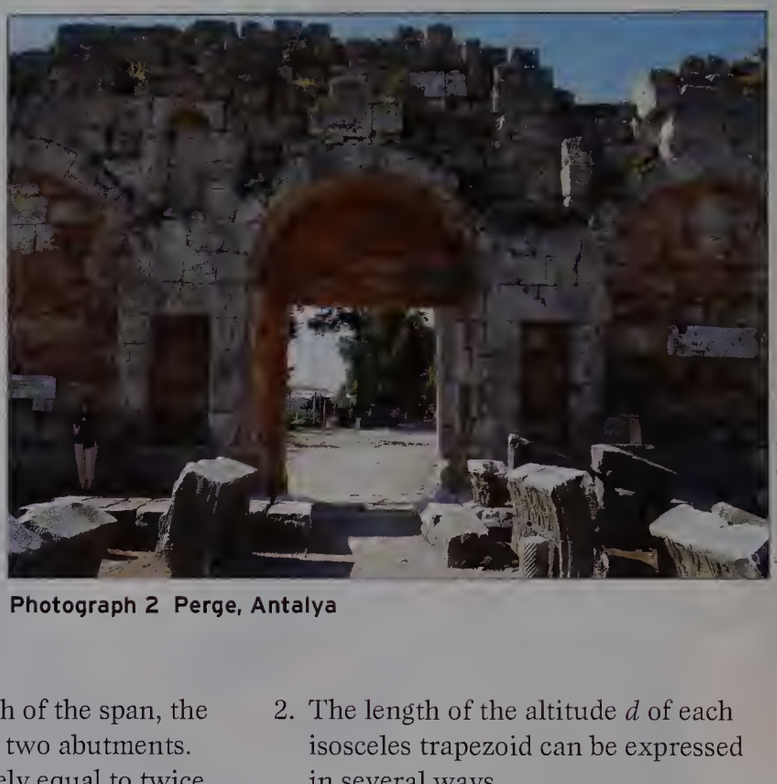
Günhan Caglayan

Archetypal Mathematics



GÜNHAN CAGLAYAN

Photograph 1 Doorway, Aspendos Antique Theater (ca. CE 180), Antalya, Turkey



GÜNHAN CAGLAYAN

Photograph 2 Perge, Antalya

The Romans used semicircular arch designs in the construction of doorways, bridges, theaters, aqueducts, and other buildings. A semicircular arch can be thought of as half of a regular polygonal ring built with *voussoirs* (wedge-like bricks) whose faces are congruent isosceles trapezoids. In a semicircular Roman arch, seen in **photograph 1**, s represents the length of the span, the distance between the two abutments. Here s is approximately equal to twice the length of the rise (r). The length of the haunch (h) is equal to the distance between the midpoint of the inner edge of the keystone and the left (or right) endpoint of the span. The angle θ is the central angle whose sides contain the two legs of each isosceles trapezoid. The number of voussoirs forming the arch (including the keystone) is represented by n. The lengths of the shorter and longer bases of each trapezoid are denoted by a and b, respectively. Finally, the leg length of each trapezoid is denoted by c.

1. Determine the measure of the central angle θ as well as the measure of each interior angle of the isosceles trapezoids in **photograph 1**. Assume that the outer edge of each trapezoid is a side of a regular polygon centered at S.
2. The length of the altitude d of each isosceles trapezoid can be expressed in several ways.
   (*a*) Express d in terms of lengths a, b, and c.
   (*b*) Express d in terms of angle θ and length c.
   (*c*) Express d in terms of θ and lengths a and b.
3. Length r in **figure 1** can also be expressed in several ways.
   (*a*) Use similar triangles to express r in terms of lengths a, b, and c.
   (*b*) Use trigonometry to express r in terms of θ.

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Edited by ***Ron Lancaster,*** *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

GÜNHAN CAGLAYAN

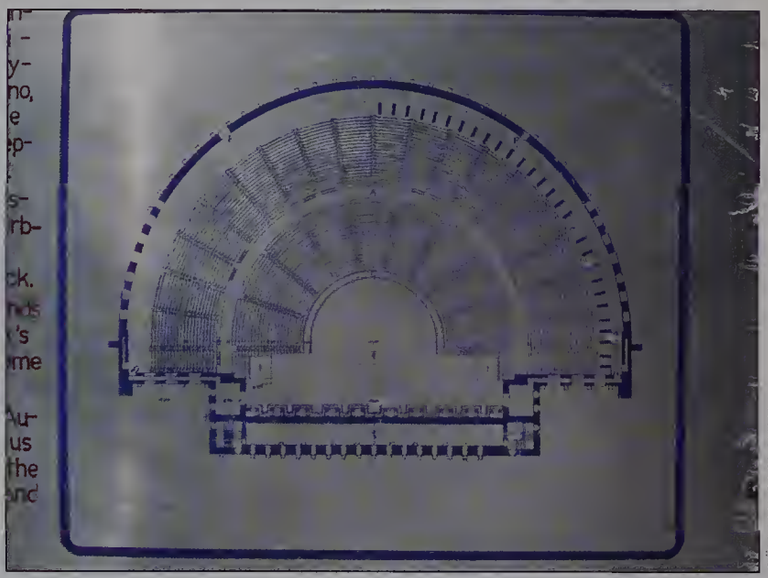
Photograph 3 Doorway, Perge Antique Theater

GÜNHAN CAGLAYAN

Photograph 4 Entrance, Phaselis, Antalya

GÜNHAN CAGLAYAN

Photograph 5 Aspendos Theater, Antalya



GÜNHAN CAGLAYAN

Photograph 6 Floor plan, Aspendos Theater

4. **Photographs 2**, **3**, and **4** show other arches in Turkey. For each, evaluate the measure of central angle θ as well as the measure of the interior angles of each isosceles trapezoid.

5. The same semicircular design can be observed in the construction of a theater's seating (see **photograph 5**). The double-layered formation in the floor plan in **photograph 6** suggests respective values of 9 and 18 for n. How do the measures of the central angles that subtend each section of seats in the inner and outer layers of the formation at Aspendos Theater relate to each other?

6. The arches are semicircular, but imagine a regular polygon inscribed in a full circle. The rise of the arch seen in **photograph 1** would be the apothem of that polygon, and the polygon's sides would be the inner edges of the voussoirs.

   (*a*) Show that $s > 2r$ and prove that

   $$\lim_{n\to\infty} \frac{s}{2r} = 1.$$

   (*b*) Explain the meaning of the limit proved in (*a*).

7. Prove that as n approaches infinity, the sum of the lengths of the inner edges of the voussoirs approaches the expression πr. Write and prove a similar expression for the sum of the lengths of the outer edges. (Hint: Use the trigonometric ratios found in problem 3[*b*].)

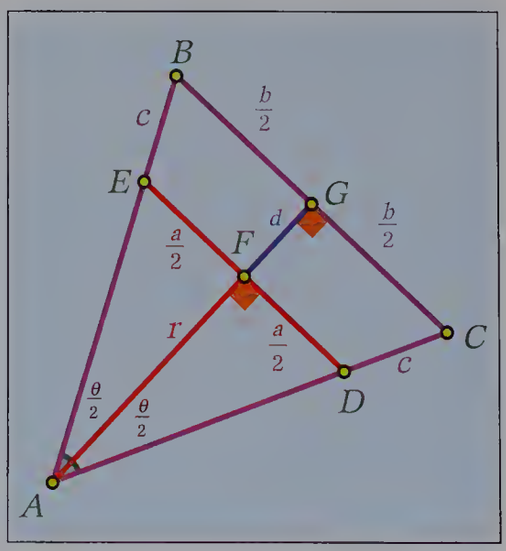


Fig. 1 The length r can be expressed in multiple ways.

MATHEMATICAL LENS solutions

1. With the regular polygon assumption, θ can be calculated as $\theta = 180°/n = 180°/11 \approx 16.4°$. As depicted in **figure 2**, ABC is an isosceles triangle because $AE = AD$ and $EB = DC$. Thus, the sum of angles B, C, and θ is 180°, leading to the following:

$$2m\angle B + \theta = 180° \rightarrow$$
$$2m\angle B = 180° - \theta \rightarrow$$
$$m\angle B = 90° - \frac{\theta}{2} \rightarrow$$
$$m\angle B = m\angle C \approx 81.8°$$

By the interior angle properties of an isosceles trapezoid, we find that

$$m\angle D = m\angle E = 180° - m\angle B$$
$$= 180° - \left(90° - \frac{\theta}{2}\right)$$
$$= 90° + \frac{\theta}{2} \approx 98.2°.$$

2. (a) In the isosceles trapezoid $BCDE$, draw the altitudes from E and D to base BC, with respective intersection points M and N, as shown in **figure 3**. Since $MN = ED = a$, $CN + BM = b - a$; and since $CN = BM$, $CN = BM = (b - a)/2$. Applying the Pythagorean theorem in either triangle BME or CND yields

$$d^2 + \left(\frac{b-a}{2}\right)^2 = c^2$$
$$d^2 = c^2 - \left(\frac{b-a}{2}\right)^2$$
$$d = \pm\sqrt{c^2 - \left(\frac{b-a}{2}\right)^2}.$$

Since d is a length, we discard the negative solution.

(b) Sin $B = d/c$, and from problem 1 we know the measure of angle B. For the arch in **photograph 1**, we can write $d \approx c \sin 81.8°$.

(c) Using another trigonometric ratio, we see that

$$\tan B = \frac{d}{\frac{b-a}{2}} \rightarrow$$
$$d \approx \frac{b-a}{2} \tan 81.8°.$$

3. (a) Triangles AFD and AGC (see **fig. 1**) are similar, implying that the corresponding sides are proportional. In particular, $AG/AF = GC/FD$. Substitution yields the following:

$$\frac{r+d}{r} = \frac{\frac{b}{2}}{\frac{a}{2}} \rightarrow 1 + \frac{d}{r} = \frac{b}{a} \rightarrow$$
$$\frac{d}{r} = \frac{b}{a} - 1 = \frac{b-a}{a} \rightarrow$$
$$\frac{r}{d} = \frac{a}{b-a}$$
$$r = \frac{ad}{b-a}$$

Now we must replace d. If we use the expression for d that we obtained in the answer to problem 2(a), r can be expressed in terms of a, b, and c as

$$r = \frac{a}{b-a}\sqrt{c^2 - \left(\frac{b-a}{2}\right)^2}.$$

The preceding equation can be written in a more compact form as

$$r = a\sqrt{\left(\frac{c}{b-a}\right)^2 - \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{\left(\frac{ac}{b-a}\right)^2 - \left(\frac{a}{2}\right)^2}.$$

This preceding equation suggests that r, $a/2$, and $ac/(b - a)$ form a Pythagorean triple and, hence, are the lengths of the sides of a

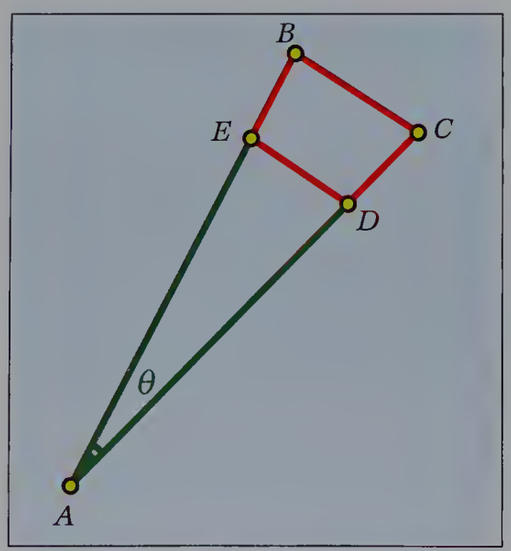


Fig. 2 Triangle ABC is isosceles.

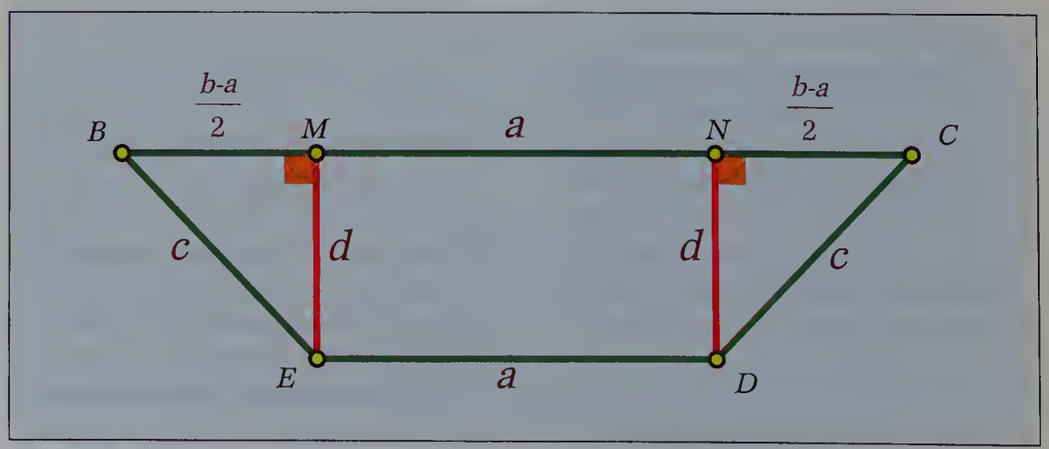


Fig. 3 Either the Pythagorean theorem or trigonometry can be used to find expressions for the altitude, d.

right triangle. Unsurprisingly, that right triangle is AFD. The quantity $ac/(b - a)$ is the length of AD. This expression for AD could have also been obtained by solving the proportion $AC/AD = GC/FD$ if the similar triangles AFD and AGC had been used.

(*b*) In the isosceles triangle ABC, the altitude drawn to the base BC bisects the vertex angle θ as well as ED and BC at F and G, respectively. Moreover, in an isosceles triangle, the bisector of the vertex angle is also perpendicular to the base. Therefore, angles AFD and AGC are right angles, meaning that AFD is a right triangle with leg lengths r and $a/2$. Using trigonometry, we can write

$$\tan\left(\frac{\theta}{2}\right) = \frac{a}{2r} \rightarrow$$

$$r = \frac{a}{2\tan\left(\frac{\theta}{2}\right)}$$

$$r = \frac{a}{2}\cot\left(\frac{\theta}{2}\right),$$

which is another way of expressing r in terms of the other variables. Since $\theta = 180°/n$, yet another expression would be

$$r = \frac{a}{2}\cot\left(\frac{180°}{2n}\right).$$

In particular, for the arch design in **photograph 1**, we find that

$$r \approx \frac{a}{2}\cot(8.8°).$$

4. In **photograph 2**, assuming that $n = 15$, we obtain $\theta = 180°/15 = 12°$. The interior angles of each trapezoid are 84° at B and C and 96° at D and E. In **photograph 3**, assuming that $n = 13$, we obtain $\theta \approx 13.8°$. The interior angles of each trapezoid are thus approximately 83.1° and 96.9°. In **photograph 4**, assuming that $n = 18$, we obtain $\theta \approx 10°$. The interior angles of each trapezoid are thus 85° and 95°.

5. **Table 1** provides the answers. The measure of the central angle for the inner layer, with $n = 9$, is twice the measure of the central angle for the outer layer, with $n = 18$.

6. (*a*) From **figure 1**, we see that AD is half the length of the span, s. Since AFD is a right triangle, $s/2 > r$, and, hence, $s > 2r$. To obtain the limit relation, we use trigonometry. In triangle AFD, we know that

$$\cos\left(\frac{\theta}{2}\right) = \frac{AF}{AD} = \frac{r}{s/2} = \frac{2r}{s}.$$

In the answer to problem 1, we learned that $\theta = 180°/n$. Substituting this expression (in radian form) in the equation $\cos(\theta/2) = 2r/s$, we obtain $\cos(\pi/2n) = 2r/s$. As n goes to infinity, $\pi/2n$ goes to 0, and therefore $\cos(\pi/2n)$ goes to 1. Thus,

$$\lim_{n\to\infty}\frac{s}{2r} = 1.$$

(*b*) Various answers are possible. For example, as n goes to infinity, the polygonal ring becomes a semicircle. Moreover, each isosceles trapezoid more closely resembles a rectangle, thus agreeing with the behavior of the variables in **table 1**.

7. The sum of the lengths of the inner edges of the voussoirs can be written as na. We begin with the trigonometric ratio obtained in the solution to problem 3(*b*), relating the variables r and a in radian form. We then multiply both sides of the equation by n and solve for na:

Table 1 Interior Angles as Functions of n and θ

n	θ	$m\angle B$, $m\angle C$ (to nearest tenth of a degree where not exact)	$m\angle D$, $m\angle E$ (to nearest tenth of a degree where not exact)
2	90	45	135
3	60	60	120
4	45	67.5	112.5
5	36	72	108
6	30	75	105
7	25.71	77.1	102.9
8	22.5	78.8	101.3
9	20	80	100
10	18	81	99
11	16.36	81.8	98.2
12	15	82.5	97.5
13	13.85	83.1	96.9
14	12.86	83.6	96.4
15	12	84	96
16	11.25	84.4	95.6
17	10.59	84.7	95.3
18	10	85	95
19	9.474	85.3	94.7
20	9	85.5	94.5

$$r = \frac{a}{2}\cot\left(\frac{\pi}{2n}\right) \to$$

$$nr = \frac{na}{2}\cot\left(\frac{\pi}{2n}\right) \to$$

$$na = \frac{2nr}{\cot\left(\frac{\pi}{2n}\right)}$$

We take the limit of the right side as n approaches infinity, using L'Hôpital's rule as needed:

$$\lim_{n\to\infty}\frac{2nr\left(\frac{b}{a}\right)}{\cot\left(\frac{\pi}{2n}\right)} = \lim_{n\to\infty}\frac{\frac{d}{dn}2nr\left(\frac{b}{a}\right)}{\frac{d}{dn}\cot\left(\frac{\pi}{2n}\right)}$$

$$= \lim_{n\to\infty}\frac{2r\left(\frac{b}{a}\right)}{\left(\frac{-\pi}{2n^2}\right)\cdot\left[-\csc^2\left(\frac{\pi}{2n}\right)\right]}$$

$$= \pi r\lim_{n\to\infty}\frac{b}{a}\frac{\sin^2\left(\frac{\pi}{2n}\right)}{\left(\frac{\pi}{2n}\right)^2}$$

$$= \pi r\lim_{n\to\infty}\frac{b}{a}\cdot\lim_{n\to\infty}\frac{\sin^2\left(\frac{\pi}{2n}\right)}{\left(\frac{\pi}{2n}\right)^2}$$

$$= \pi r\cdot 1\cdot 1 = \pi r$$

This result suggests that as n approaches infinity, the polygonal ring becomes more like a semicircle. As shown below, as n approaches infinity, the sum of the lengths of the outer edges of the voussoirs approaches the same expression: πr.

The sum of the lengths of the outer edges of the voussoirs can be written as nb. Referring to **figure 1**, we can relate the variables r and b using trigonometry:

$$\frac{r+d}{\frac{b}{2}} = \cot\left(\frac{\pi}{2n}\right) \to$$

$$r + d = \frac{b}{2}\cot\left(\frac{\pi}{2n}\right)$$

Recall that in problem 1, we established that d can be expressed in terms of r as $d = r(b/a - 1)$. Once we make this substitution, we obtain the following:

$$cr + r\left(\frac{b}{a} - 1\right) = \frac{b}{2}\cot\left(\frac{\pi}{2n}\right) \to$$

$$r\cdot\frac{b}{a} = \frac{b}{2}\cot\left(\frac{\pi}{2n}\right) \to$$

$$nr\cdot\frac{b}{a} = \frac{nb}{2}\cot\left(\frac{\pi}{2n}\right) \to$$

$$nb = \frac{2nr\cdot\frac{b}{a}}{\cot\left(\frac{\pi}{2n}\right)}$$

If we take the limit of both sides as n approaches infinity, again using L'Hôpital as needed, we obtain the following, as desired:

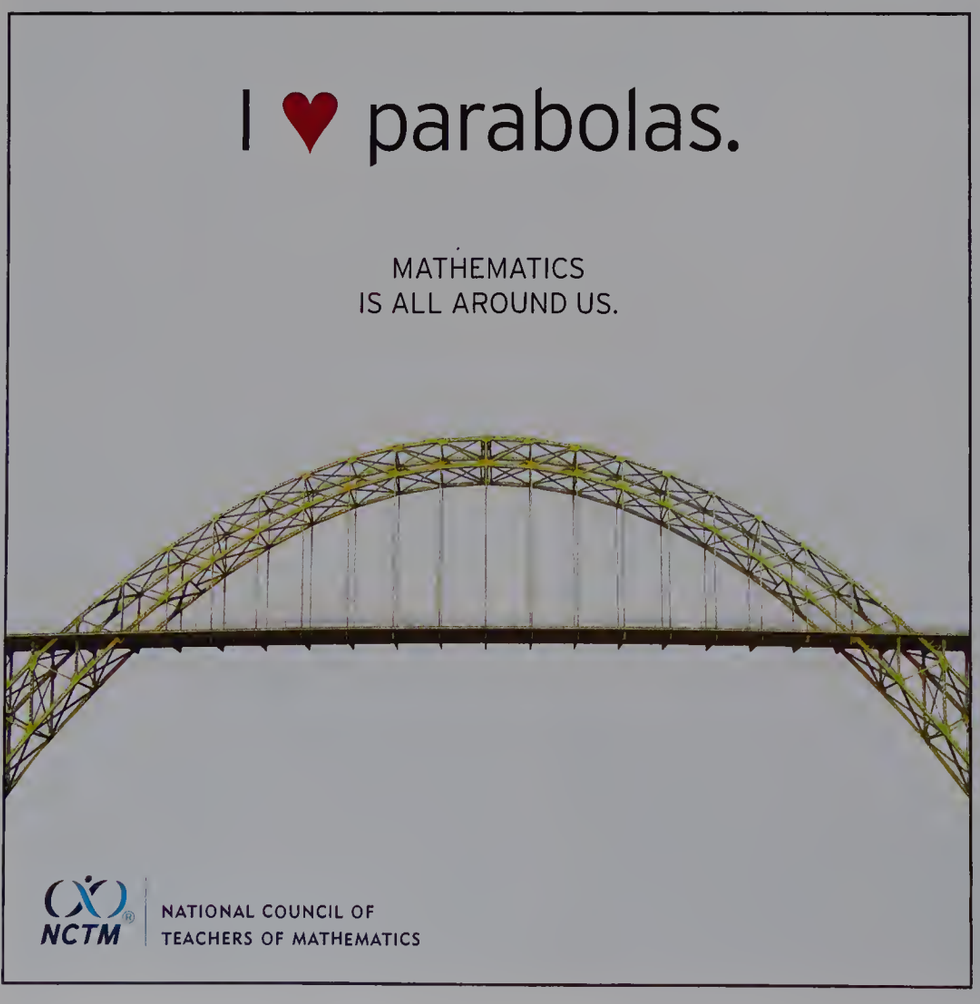
$$\lim_{n\to\infty}\left(\frac{2nr\left(\frac{b}{a}\right)}{\cot\left(\frac{\pi}{2n}\right)}\right) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}2nr\left(\frac{b}{a}\right)}{\frac{d}{dn}\cot\left(\frac{\pi}{2n}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2r\left(\frac{b}{a}\right)}{\left(\frac{-\pi}{2n^2}\right)\left(-\csc^2\left(\frac{\pi}{2n}\right)\right)}\right)$$

$$= \pi r\lim_{n\to\infty}\left(\frac{b}{a}\cdot\frac{\sin^2\left(\frac{\pi}{2n}\right)}{\left(\frac{\pi}{2n}\right)^2}\right)$$

$$= \pi r\lim_{n\to\infty}\left(\frac{b}{a}\right)\cdot\lim_{n\to\infty}\left(\frac{\sin^2\left(\frac{\pi}{2n}\right)}{\left(\frac{\pi}{2n}\right)^2}\right)$$

$$= \pi r\cdot 1\cdot 1 = \pi r$$



GÜNHAN CAGLAYAN, gcaglaya@fiu.edu, teaches mathematics at Florida International University in Miami. He is interested in preservice mathematics teacher education and student learning through modeling and visualization.

more 4U

For a photograph for your own mathematical questions, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt039**.

CALL FOR MANUSCRIPTS

2014 Focus Issue

What Makes Ideas Stick?

Teachers design lessons to engage students with big ideas of mathematics. Does engagement necessarily imply that students will retain what we hope they will retain? How do students in your classes develop an appreciation for mathematical big ideas and the relationships among them? What lesson characteristics seem to promote the permanence of concepts in the minds of your students?

What works? Write about your big ideas—*big* big ideas or *small* big ideas—that help make mathematics stick. Following are a few suggestive, but not prescriptive, prompts:

- Great lessons stay in the minds of students. What lessons do you have that engage your students in exciting and meaningful mathematical experiences?
- Some ideas reside with students because of the instruction that introduces them more than the inherent mathematics. Do you start with an intriguing idea and then develop it throughout class, or do the big ideas build in the background?
- Students often surprise us with their own "aha!" moments, usually as they experience the epiphany of an unexpected connection. What are these moments, and how do we foster these connections and epiphanies?
- How do reasoning and sense making evolve to foster mathematical memory? What are high-leverage practices that help ideas stick?
- What kinds of challenges do big mathematical ideas face in the current political climate?
- How can technology impact the "stickiness" of ideas?
- How do we ensure that students internalize the mathematics that they need to be successful in subsequent courses? How do they develop the mathematical sophistication to adapt to an ever-changing world?

Contribute to this special *MT* issue by highlighting mathematical and pedagogical big ideas from your own teaching. Show us what helps students retain ideas and what leads them to an appreciation of broad concepts.

The *MT* Editorial Panel invites submissions for the 2014 Focus Issue. Please submit manuscripts at **mt.msubmit.net** by **May 1, 2013**. Be sure to check the box indicating that this manuscript is for the 2014 Focus Issue entitled What Makes Ideas Stick? Guidelines for the preparation of manuscripts can be found at www.nctm.org/publications/content.aspx?id=22602. No author identification should appear in the manuscript's article file. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

JOKMEDIA/ISTOCKPHOTO.COM

MATHEMATICS teacher

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Galileo, Gauss, and

Using ideas of Galileo and Gauss but avoiding calculus, students create a model that predicts whether a fly ball will clear the famous left-field wall at Fenway Park.

Dan Kalman and Daniel J. Teague

Presented for your consideration: Galileo, Gauss, and the Green Monster. Each is legendary in his—or its—own way.

Wasn't Galileo the fellow who dropped cannonballs from the leaning tower of Pisa to demonstrate something about falling bodies? Then there is Gauss, a giant of mathematics and physics who made unparalleled contributions to both fields. But he may be most famous for a feat performed in the fifth grade. More contemporary (and not a person), the Green Monster is the left-field wall at the home

the Green Monster

of the Boston Red Sox, Fenway Park. Measuring 37 feet and 2 inches and standing a little more than 300 feet from home plate, it has gobbled up many a line drive that, in another ballpark, might have been a home run.

So, three icons. But what do they have to do with one another? In answer, we develop projectile motion from scratch. We hope that this development provides new insights and deeper understanding for teachers as well as opportunities for enrichment activities for students.

GAUSSIAN ADDITION

According to legend, Carl Friedrich Gauss (1777–1865) and his fellow students were given the task of adding the whole numbers from 1 to 100. In some tellings, the task was assigned as punishment for misbehavior. But all versions agree that Gauss immediately wrote 5050 on his slate and handed it to his teacher. Assuming more mischief, the teacher initially responded angrily but soon enough realized that Gauss had indeed computed the sum correctly.

JOE ROBBINS/GETTY IMAGES

Students today can reconstruct the sort of reasoning that Gauss might have used. List the numbers from 1 to 100 in a single long line and then add them from the ends inward. We add first 1 and 100, then 2 and 99, then 3 and 98, and so on. Each of these additions produces 101, and we have 50 additions because the innermost pair is 50 + 51. The total is thus $101 \cdot 50 = 5050$. This technique can be used to add any number of consecutive integers or, more generally, the terms of any arithmetic sequence. In particular, if we sum all the whole numbers from 1 to n, the result will be $n(n + 1)/2$. Once students know the "trick," it is not so mysterious, but for those who don't, this is an amazing shortcut.

An algebraically simpler result holds for summing just the odd integers, reaching a total of n^2 when n terms are added. For example, to add the first 10 odd integers—

1 3 5 7 9 11 13 15 17 19

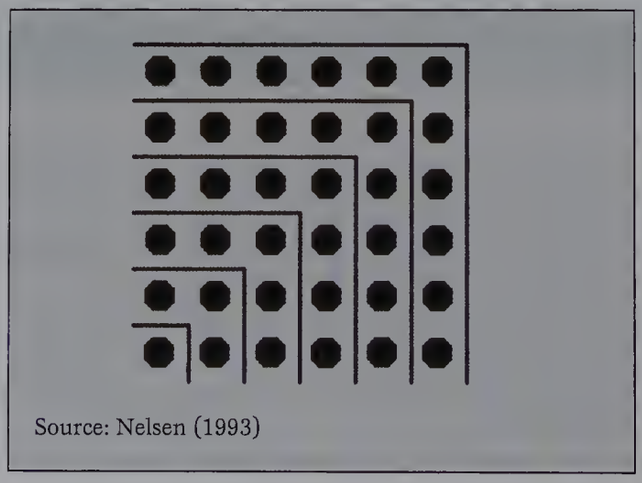
—add again, from the outside in, 1 + 19, 3 + 17, and so on. Each time we get a total of 20 (or $2n$), and we have 5 (or $n/2$) such pairs. Thus, the total is $100 = n^2$.

When asked to find the sum of the first n odd positive integers, students will use many approaches. In addition to the numerical and algebraic methods that students most readily apply, a beautiful geometric demonstration was devised by Nicomachus, a follower of Pythagoras (see **fig. 1**).

By whatever route derived, the result that the first n odd counting numbers sum to n^2 is surely a memorable one. We will soon see that this is the perfect tool for applying Galileo's discoveries to the conquest of the Green Monster.

FALLING FOR GALILEO

Sums of odd numbers arise naturally in Galileo Galilei's (1564–1642) investigations of falling bodies.



Source: Nelsen (1993)

Fig. 1 About CE 100, Nicomachus, a follower of Pythagoras, devised this method for adding consecutive odd integers.

There he argued that the motion of a falling body must be similar to that of a ball rolling down an inclined plane. The details of his argument will be too great a digression here, but the key ideas involve properties that are unchanged by the angle of inclination (Fowler 2009). Galileo's analysis includes the following conclusion: "... the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity" (Galileo 1638, 1914) (see **fig. 2**).

This same principle applies to a ball rolling on an inclined plane. For example, if a ball happens to roll 5 feet in the first 2 seconds, then it will roll $3 \cdot 5 = 15$ feet in the next 2 seconds, $5 \cdot 5 = 25$ feet in the third 2 seconds, and so on. What is the total distance covered in, say, 30 seconds? That would be 15 2-second periods, so the sum of the distances would be $1 \cdot 5 + 3 \cdot 5 + 5 \cdot 5 + \cdots$, for a total of 15 terms. If we factor out the common 5, what remains is the sum of the first 15 odd integers. Using Gauss's trick, we know that this sum will be 15^2. So the distance traveled in 30 seconds will be $5 \cdot 15^2 = 1125$ feet. Similarly, we can see that the ball will roll $5 \cdot n^2$ feet in n 2-second periods.

Galileo's principle can also be used to find distance traveled in shorter time periods. In the example we have been considering, the ball travels 5 feet in the first 2 seconds. How far did it travel in the first second? Perhaps half the distance, or 2.5 feet. But then, according to Galileo's principle, if the ball rolls 2.5 feet in the first second, it must roll 3 times as far, 7.5 feet, in the next second. This is not possible, given that the ball rolls only 5 feet in the first 2 seconds. By trial and error, we could make successive guesses at the correct distance that the ball rolls in the first second. A more direct approach uses algebra. If the ball travels x feet in the first second, then it should travel $3x$ feet in the next second, implying a total of $4x$ feet for 2 seconds. But we know that the ball travels 5 feet in 2 seconds. Therefore, we conclude that $4x = 5$, giving x as 1.25 feet.

This style of reasoning can be pushed a bit further to determine the distance traveled by the ball in any rational amount of time. But rather than follow this thread for the rolling ball, let's consider instead a falling body. Today it is common knowledge that, as a good first approximation, a falling body will travel about 16 feet in the first second. Using the same logic as before, we can deduce how far it would fall in time t for any rational (i.e., fractional) amount of time.

As an example, take $t = 19/7$. How far will a body falling from rest travel in 19/7 seconds? As a first step, we can find the distance that it falls in 1/7 second. For the moment, take that distance to

be x. The body will fall x feet in the first 1/7 second, $3x$ feet in the next 1/7 second, $5x$ feet in the next, and so on. We see that the total distance that it falls in 1 second would be

$$x + 3x + 5x + 7x + 9x + 11x + 13x$$
$$= (1 + 3 + 5 + 7 + 9 + 11 + 13)x.$$

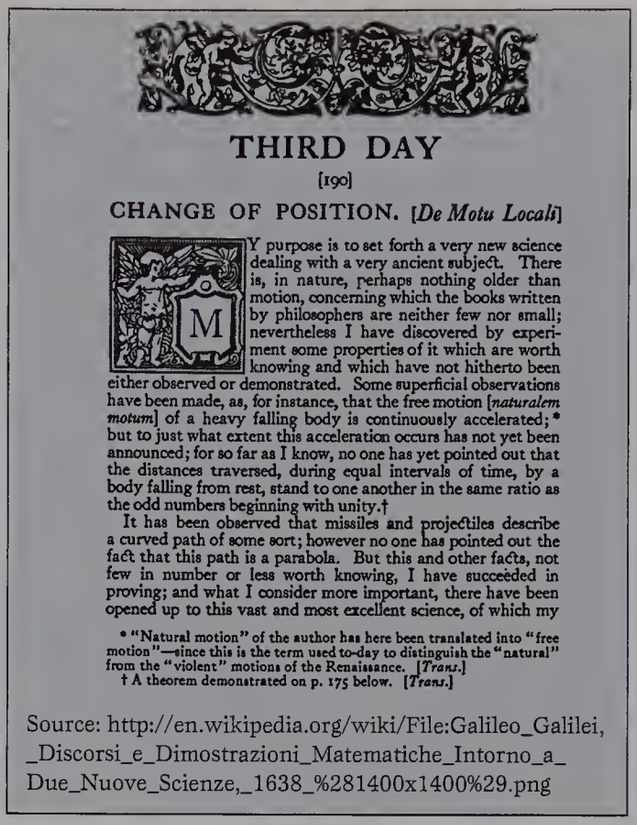
Using Gauss's trick, we find that the first 7 odd integers add to 7^2, so the distance traveled is 7^2x. But we already know that a falling object travels 16 feet in the first second. Equating these two quantities shows that $x = 16/7^2$.

Now we can complete the original question. We want to find the distance that the body falls in 19 equal 1/7-second time periods. Reasoning as before, we find that the answer must be 19^2 times the distance fallen in the first of the time periods. This computation gives $19^2(16/7^2) = 16(19/7)^2$ feet fallen in 19/7 seconds. And the method of derivation shows that the same pattern would hold for any other fraction. In summary, in any rational amount of time t, an object falling from rest must travel $16t^2$ feet, a familiar result for modern readers, derived here without calculus.

PARABOLIC MOTION

It is two short steps from the analysis above to the general idea of parabolic trajectories. First, we have to extend our ideas to objects that do not start out at rest; then we take into account the possibility of nonvertical motion. Regarding the first step, we adopt an additional assumption—that free-falling motion is symmetric. To be more specific, imagine throwing an object vertically upward and filming it as it climbs to a high point and then drops back into your hand. The idea of symmetry that we have in mind is this: The film will look the same running forward or backward. In this case, we can chart the motion of the falling body from the high point either forward or backward in time, using the same computations. And because the body is at a standstill when it reaches its zenith, both the forward and reverse time trajectories are described by the equation derived earlier for a body falling from rest.

To illustrate how this works, let's derive an equation for the height as a function of time. Here, the time variable t can be likened to the reading on a running stopwatch. We will not be concerned with the absolute time when the watch was started but, rather, will measure elapsed time from that point forward. Now suppose that we toss a ball vertically upward and that it reaches a maximum height of 155 feet when $t = 3$. At the high point, the body is momentarily at a standstill, so its motion afterward is that of a body falling from rest. In particular, at any time $t > 3$, the ball will have fallen



THIRD DAY

[190]

CHANGE OF POSITION. [*De Motu Locali*]

MY purpose is to set forth a very new science dealing with a very ancient subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated. Some superficial observations have been made, as, for instance, that the free motion [*naturalem motum*] of a heavy falling body is continuously accelerated;* but to just what extent this acceleration occurs has not yet been announced; for so far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity.†

It has been observed that missiles and projectiles describe a curved path of some sort; however no one has pointed out the fact that this path is a parabola. But this and other facts, not few in number or less worth knowing, I have succeeded in proving; and what I consider more important, there have been opened up to this vast and most excellent science, of which my

* "Natural motion" of the author has here been translated into "free motion"—since this is the term used to-day to distinguish the "natural" from the "violent" motions of the Renaissance. [*Trans.*]

† A theorem demonstrated on p. 175 below. [*Trans.*]

Source: http://en.wikipedia.org/wiki/File:Galileo_Galilei,_Discorsi_e_Dimostrazioni_Matematiche_Intorno_a_Due_Nuove_Scienze,_1638_%281400x1400%29.png

Fig. 2 In his *Dialogues Concerning Two New Sciences* (p. 153), Galileo sets forth his ideas about *naturalem motum*, translated by Crew and de Salvio (1914) as *free motion*.

for $t - 3$ seconds. According to our prior results, in that time the ball drops $16(t - 3)^2$ feet and so is at a height

$$h(t) = 155 - 16(t - 3)^2, \text{ for } t \geq 3.$$

The symmetry assumption says that this equation will also be valid for $t < 3$. For example, consider $t = 2$. With 1 second to go before reaching the high point, how high is the ball? We are assuming that the film looks the same in forward and reverse, so the distance that the ball climbs from time 2 to 3 is exactly the same as the distance that it falls from time 3 to 4, so $h(2) = h(4)$. And that is what we obtain by letting $t = 2$ in our equation. In this way, we see that the equation for h will be valid for any time t as long as the ball is in the air at that time.

In the general case, if we denote the maximum height by h_{top} and the time when the high point is reached by t_{top}, the equation becomes

$$h(t) = h_{top} - 16(t - t_{top})^2.$$

Algebraic simplification gives this equation in the form $h(t) = -16t^2 + bt + c$ where $b = 32t_{top}$ and $c = h_{top} - 16t_{top}^2$. Here we are interested mainly in recognizing that $h(t)$ is a quadratic function. The equations for the coefficients b and c will not be important for what follows, although it will be significant that c has the alternate interpretation as the height

at time $t = 0$. Later we will see a useful alternative interpretation for b as well.

Now that we have extended the result for a falling body to any vertical trajectory, let's proceed to the second variation: allowing the body to move horizontally as well as vertically. Again, Galileo is our guide. From his experiments with balls rolling off the edge of a horizontal plane and then falling freely, he argued that horizontal motion and vertical motion can simply be superimposed. Moreover, for a falling body traveling both vertically and horizontally, although gravity will affect the vertical motion, the horizontal motion continues at a constant rate. If this rate is 30 feet per second, say, then from the starting point the horizontal position at time t will be $30t$ feet.

Now let's represent the location of the body on a two-dimensional graph, taking x to be the horizontal position and y to be the vertical position (what we previously referred to as height h). We have equations for each of these variables:

$$x = 30t$$
$$y = -16t^2 + bt + c$$

Solving for t in the first equation and substituting in the second produces the following:

$$y = -(16/900)x^2 + (b/30)x + c \qquad (1)$$

We therefore see that the trajectory of a free-falling body is a parabola. Again, this result is no surprise to modern readers. But it is gratifying to come so far without the methods of calculus. And experience has shown that students enjoy seeing how the tools of algebra and a few simple (and plausible) assumptions lead to such detailed conclusions about the paths of free-falling bodies.

ESCAPING THE GREEN MONSTER

At last we come to the Green Monster. Using quadratic equations to analyze parabolic trajectories is so familiar in secondary school mathematics that determining whether a fly ball will clear the Green Monster is undoubtedly a pedestrian exercise. However, the traditional analysis depends on this equation—

$$h(t) = -16t^2 + (\textit{initial vertical velocity})t + \textit{initial height}$$

—which, in turn, depends on calculus, which we prefer to avoid. Besides, the idea of a hitter carefully controlling the initial vertical velocity of a batted ball is not very plausible. The batter more likely can control the direction in which the ball is hit. How does this idea relate to the ball's trajectory? We again adopt a reasonable hypothesis: A straight line drawn from the point of impact in the initial direction of motion will be tangent to the trajectory.

Mention of *tangent* tempts us to apply calculus. But for a parabola, the tangent can be found by more elementary means. For concreteness, we assume that the ball is on the y-axis at $(0, c)$ when struck. Thus, the trajectory has y-intercept c and equation $y = ax^2 + bx + c$. Any straight line through the same intercept will have equation $y = mx + c$. Combining the two equations, we can find where the line and the parabola meet. Equating the right sides of the equations leads to

$$ax^2 + bx = mx,$$

or, more simply,

$$x(ax + b - m) = 0.$$

One solution, $x = 0$, corresponds to $(0, c)$, where we already know that the line and the parabola intersect. The other solution, $x = (m - b)/a$, will generally represent a second point of intersection. However, when $m = b$, the two intersections coincide. In this case, the line $y = mx + c$ is tangent to the parabola, establishing an elegant general result:

> For the parabola $y = ax^2 + bx + c$, the tangent line at the y-intercept is given by the equation $y = bx + c$. (2)

None of the students that we observed considered either the geometry of the problem (noting that if $x = 0$ is the unique solution to a quadratic equation, then the only possible form of the quadratic is $y = ax^2$, so $m = b$) or the algebraic fact that a quadratic has a unique zero only in the case of a double root and, hence, a zero discriminant. This observation provides another derivation of $m = b$, echoing one of Descartes's methods for finding tangent lines.

Restating (1) in a slightly different form will make it more readily applicable to the problem of escaping the Green Monster. Assume that the batter can hit the ball along a line with a specified elevation angle θ and, thus, with a slope of $\tan\theta$. The equation of the ball's trajectory must then have the form $y = ax^2 + x\tan\theta + c$.

Now we can tackle that monstrous wall. Suppose that a batter can hit a ball at a speed of 150 feet per second (a little more than 100 miles per hour). Will a ball struck 3 feet above the ground and traveling initially along a line with elevation 20° go over the Green Monster?

The trajectory of the batted ball has equation $y = ax^2 + (\tan 20°)x + 3$. As in our earlier analysis

and equation (2), the coefficient of the quadratic term is $-16/k^2$ where k is the horizontal speed of the ball. Because the ball is initially traveling 150 feet per second on an incline at 20°, its horizontal speed is $k = 150\cos 20°$. Therefore,

$$\frac{-16}{k^2} = \frac{-16}{(150\cos 20°)^2},$$

which leads to

$$y = \frac{-16x^2}{(150\cos 20°)^2} + (\tan 20°)x + 3.$$

The distance from the plate to the Green Monster is roughly $x = 300$ feet. Substituting, we find that

$$y = \frac{-16(300)^2}{(150\cos 20°)^2} + (\tan 20°)300 + 3 = 39.7\text{ft}.$$

So the ball clears the 37-foot, 2-inch wall by a little more than 2 feet. A particularly nice student write-up of this final step is shown in **figure 3**.

We have used this problem with second-year algebra and precalculus students. The work shown comes from a precalculus class; the algebra students needed more direction and support as they worked through the problem. Students with whom we have used this problem have been impressed by the number of mathematics topics incorporated in the problem and how they needed to use their knowledge flexibly. Thus, the problem makes an excellent summation of the course content toward the end of the term.

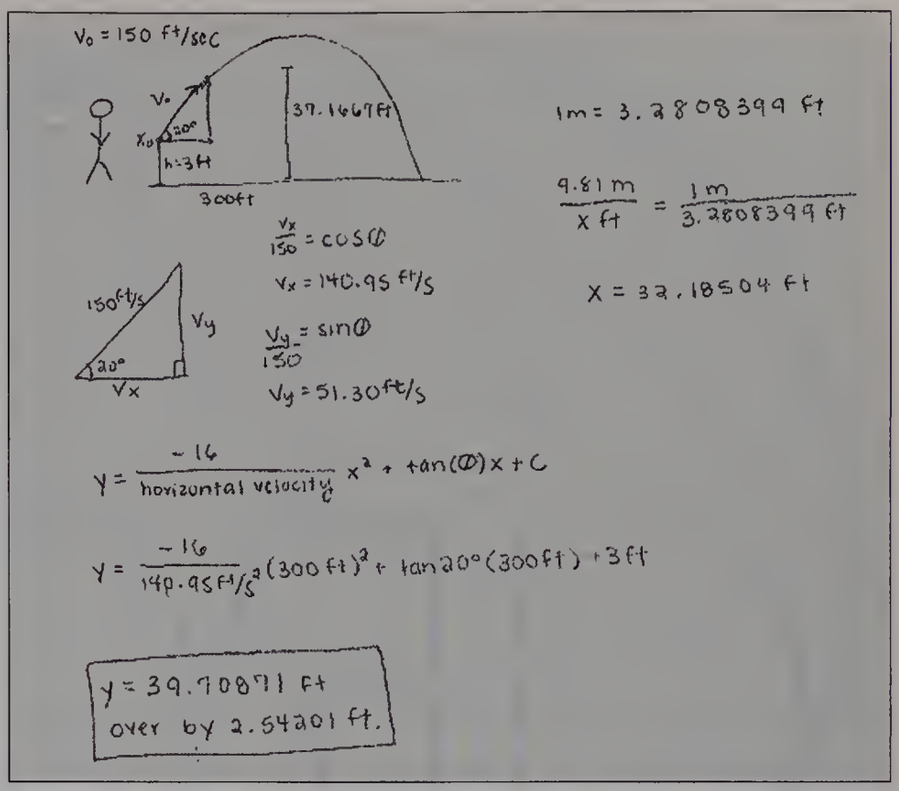


Fig. 3 A precalculus student completed this lovely solution.

POSTSCRIPT: A MIRACLE OF CALCULUS

Historically, the methods applied here predate calculus. In particular, we found a tangent line using essentially an approach of Descartes, developed about thirty years before the work of Newton and Leibniz (Suzuki 2005). We have seen that very simple mathematical and physical hypotheses lead, by elementary methods, to fairly sophisticated conclusions, but our success depends heavily on the role played by quadratic functions in the problem we considered. Applying a similar approach for more sophisticated models or more complicated systems would be far more difficult, perhaps even impossible.

In contrast, the tools of calculus, which provide an even simpler route to the conclusions we have reached, are also much more broadly applicable. Properly understood, this is rather astonishing. Calculus was an amazing advance in mathematics, not just because its methods are remarkably simple and not just because they are incredibly powerful but because they are both at the same time. More calculus students should have an opportunity to understand this.

REFERENCES

Fowler, Michael. Fall 2009. "Galileo's Acceleration Experiment." http://galileoandeinstein.physics.virginia.edu/lectures/gal_accn96.htm.

Galileo, Galilei. (1638) 1914. *Dialogues Concerning Two New Sciences*. Translated by Henry Crew and Alfonso de Salvio. New York: Macmillan.

Nelsen, Roger B. 1993. *Proofs without Words: Exercises in Visual Thinking*. Washington, DC: Mathematical Association of America.

Suzuki, Jeff. 2005. "The Lost Calculus (1637–1670): Tangency and Optimization without Limits." *Mathematics Magazine* 78 (5): 339–53.

DAN KALMAN, kalman@american.edu, teaches mathematics at American University and formerly worked as an applied mathematician in the aerospace industry. He enjoys algebra, analysis, and alliteration. DANIEL J. TEAGUE, teague@ncssm.edu, teaches at the North Carolina School of Science and Mathematics in Durham. He is interested in mathematical modeling and finding problems that connect concepts from different areas of mathematics.

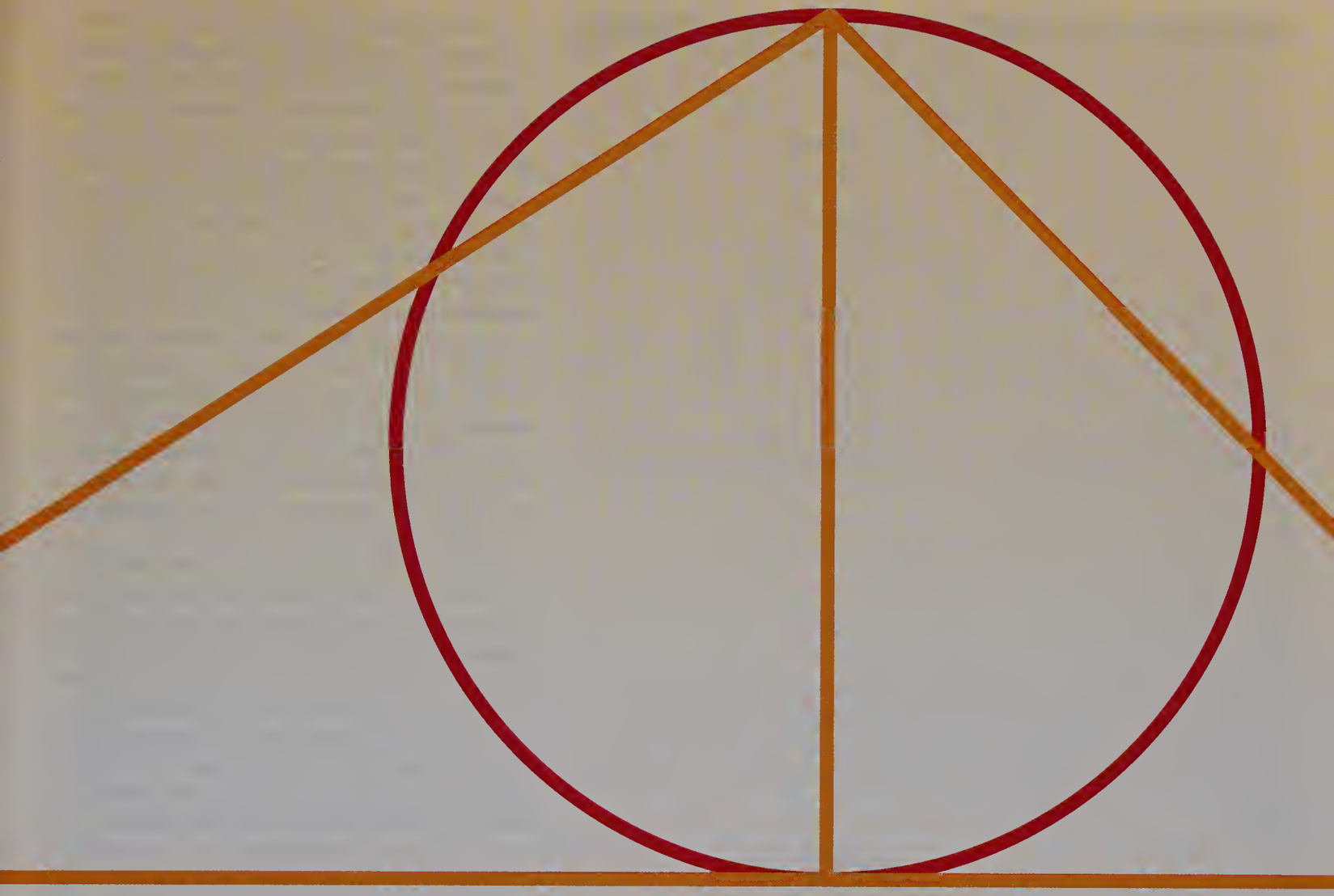
GEOMETRIC REASONING ABOUT A CIRCLE PROBLEM

An open-ended problem about a circle illustrates how problem-based instruction can enable students to develop reasoning and sense-making skills.

Gloriana González and Anna F. DeJarnette

What does problem-based instruction do for students and teachers? The open-ended geometry problem presented here along with examples of students' work on the problem illustrates how problem-based instruction can help students develop their mathematical proficiency.

NCTM's *Principles and Standards for School Mathematics* (2000) has promoted the use of problem-based instruction for teaching mathematics with understanding (p. 52). Problem-based instruction provides opportunities for students to develop their reasoning and sense-making skills (NCTM 2009). Recent studies have shown that students who experience problem-based instruction improve their perceptions about mathematics and about themselves as mathematical learners (Ridlon 2009). Moreover, instruction that incorporates problem-based lessons does not compromise students' development of the mathematical skills that are promoted when using traditional curricula. At the same time, problem-based instruction pushes students to use multiple problem-solving strategies



and to work on more cognitively demanding tasks (Cai et al. 2010; Ridgway et al. 2003). However, some teachers still have difficulty understanding how to make problem-based instruction a reality in their classrooms.

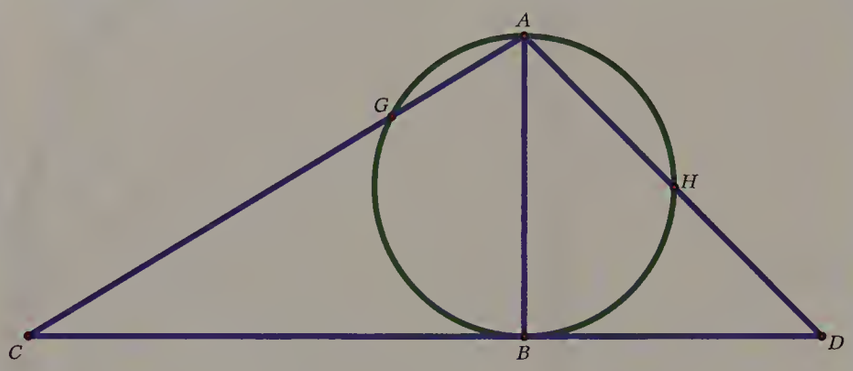
In this article, we show an example of a problem-based lesson taught in a geometry classroom—the circle problem (see **fig. 1**)—and provide evidence of geometry students' reasoning and sense making when working on it. The circle problem provided a context for students to develop competency in the five strands of mathematical proficiency outlined in *Adding It Up*: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, and Findell 2001).

Our analysis of students' work focuses on the latter three competencies, which are less commonly discussed in the literature. *Strategic competence* refers to the ability to formulate, represent, and solve mathematical problems; *adaptive reasoning* is the capacity for logical thought, reflection, explanation, and justification; and *productive disposition* refers to the inclination to see mathematics as worthwhile combined with a belief in diligence and one's own efficacy (Kilpatrick, Swafford, and Findell 2001, p. 116). By developing insight into how students can build mathematical proficiency through specific problem-based lessons, teachers may become more equipped to teach mathematics through problems.

The circle problem was used in two sections of a regular high school geometry class in a large urban midwestern school. The first author, in collaboration with a teacher whom we call Mr. Brown, developed a lesson plan with the circle problem (see **fig. 2**), which is based on an exercise in a traditional geometry textbook (Moise and Downs 1975, p. 488). The variation of the problem was innovative because students could perceive the problem as one of numerical calculation; on inspection, however, much more than performing calculations was required. Moreover, the problem anticipated a theorem that students would study soon, thus instilling in them the need to prove that theorem.

The problem was given at the end of the year, when students were studying a unit on circles. At

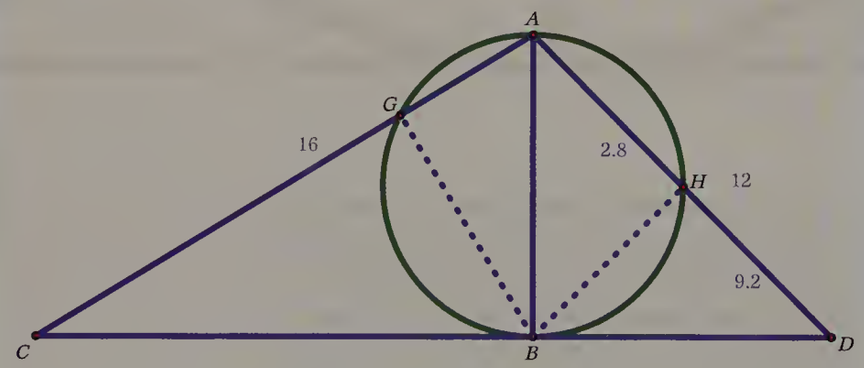
that point, the class had already covered the mathematical concepts relevant for solving the problem—for example, the Pythagorean theorem, similar triangles, theorems about the geometric mean in right triangles, and trigonometry.



AB is a diameter. *CD* is tangent to the circle at *B*. *AC* = 16 cm, *AD* = 12 cm, and *AH* = 2.8 cm. Find as much information as possible about the segments and the angles in the diagram.

Fig. 1 Students worked in groups to solve the circle problem.

In the diagram below, $\overline{AB}$ and $\overline{CD}$ are perpendicular. Adding the auxiliary lines *BH* and *BG*, we notice that angles *AHB* and *AGB* intercept the diameter; so apply the theorem about inscribed angles to conclude that angles *AHB*, *DHB*, *AGB*, and *CGB* are all right angles.



Notice that *BH* is the geometric mean of *AH* and *HD*. Therefore, we can set up the ratio

$$\frac{2.8}{BH} = \frac{BH}{9.2}$$

to find that $BH \approx 5.08$, where we round to the nearest hundredth. Now, use the Pythagorean theorem in triangles *BHD* and *ABD* to find that $BD \approx 10.51$ and $AB \approx 5.79$. Knowing *AB*, we can compute that $CB \approx 14.92$. Now we can consider triangle *ABC* and notice that *BC* is the geometric mean of *CG* and *CA*. So we set up the ratio

$$\frac{x}{14.92} = \frac{14.92}{16}$$

and find that $CG \approx 13.91$. Therefore, $GA \approx 2.09$.

Fig. 2 One solution involved using the geometric mean.

The geometric mean occurs in two theorems about right triangles. One theorem states that the measure of the altitude to the hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse; the other states that the measure of a leg is the geometric mean between the measures of the hypotenuse and the segment of the hypotenuse adjacent to that leg.

Students had not yet learned the secant-tangent product theorem, so the problem served as an introduction to that theorem and the techniques necessary for proving it. Thus, the circle problem gave students the opportunity to review theorems that they had previously studied and to apply those theorems in a new context. Because the circle problem was designed to allow students to develop their own strategies, it did not name any theorems explicitly, and students had to recall relevant knowledge to solve the problem.

Mr. Brown allowed students to work on the problem for two days. During day 1, the problem was introduced, and then students worked in small groups of three or four for the entire class period. In his introduction, Mr. Brown mentioned theorems about circles that students had studied recently but did not mention concepts that they had studied earlier in the year and that were fundamental for solving the problem, such as similarity, the Pythagorean theorem, or the geometric mean. During day 2, Mr. Brown led a whole-class discussion of the solution of the problem. Our analysis is based on videos from day 1, while students were working on the problem in small groups, and worksheets from students in one class. From these we show how the problem enabled students to build strategic competence, adaptive reasoning, and a productive disposition.

BUILDING STRATEGIC COMPETENCE

One benefit of problem-based mathematics instruction is that it allows students to decide what the problem is and the best way to solve it. With the circle problem, Mr. Brown did not give students specific guidelines about how to solve it. Instead, he told them to find as much information about the diagram as possible, and they had to decide on their own what strategies they should apply to solving the problem.

As a result, students applied various strategies. In our analysis of twenty-two students' worksheets from one class, we found that nearly all students used multiple strategies to work on the problem, relying on key features of the diagram. **Table 1** lists all the strategies that students used and the number of students who demonstrated evidence of using each strategy. The twenty-two students whose work we analyzed used a total of sixty strategies; thus, on average, each student used about three different strategies to work on the problem.

Students' use of the various strategies for working with the diagram, such as adding auxiliary lines and marking right angles on the diagram, is evidence of their strategic competency because the diagram provided did not include everything that they needed to solve the problem. For example, the problem said that segment AB was a diameter and that segment CD was a tangent. However, the given diagram did not include markings for denoting that $\overline{AB}$ and $\overline{CD}$ were perpendicular. Therefore, students would have to identify relevant information in the problem and add the relevant markings to the diagram.

By providing opportunities for students to add markings to the diagram, Mr. Brown made students accountable for recalling prior knowledge necessary for solving the problem and allowed them to integrate information about a geometric figure into the diagram. Because students were expected to draw auxiliary lines (without a cue that such lines were needed), they had to reason about strategies that could be helpful for solving the problem by transforming the diagram and also about the justification for adding such lines.

DEVELOPING ADAPTIVE REASONING

The circle problem offered multiple starting points and solution strategies for students. From our analysis of students' worksheets, we learned that students drew on their prior knowledge about different geometric concepts to solve the problem. One widely-used strategy was to apply the Pythagorean theorem to find the side lengths of right triangles. Identifying and applying similar triangles and the geometric mean was a less common approach, although using at least one of these techniques was essential to solve the problem completely.

Figure 3 shows a summary of the prior knowledge that students drew on, based on our examination of the twenty-two worksheets. For example, six students used the Pythagorean theorem and similar triangles; three students used both those concepts as well as the geometric mean; and two students used only the Pythagorean theorem. In their use of multiple strategies to solve the problem, students showed their capacity for logical thought about how certain elements of the problem were related and also their ability to justify their computations.

Students' choice of using similar triangles or the geometric mean to solve the problem is an example of adaptive reasoning. They applied their knowledge about specific theorems and properties of geometric objects to the specifics of the circle problem. After drawing auxiliary segment BH in the diagram, students could find BH either by identifying similar triangles or by applying the geometric mean.

Table 1 Strategies for Working with the Diagram

Strategy	Number of Students Who Displayed Use of This Strategy
Adding auxiliary lines	21
Marking right angles on the diagram	21
Redrawing parts of the diagram	8
Changing the orientation of parts of the diagram	8
Marking congruent pieces on the diagram	2

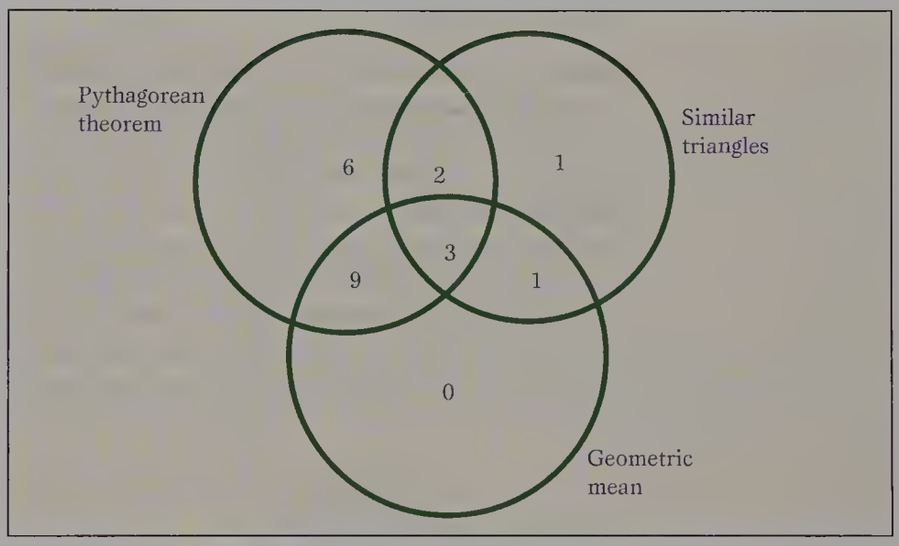


Fig. 3 Students used some or all of these concepts in their solutions.

Research on learning suggests that learners' ability to organize their prior knowledge so that they can make adaptations to new situations is crucial for learning (NRC 2000, pp. 45–48). Therefore, a problem that enables students, on their own, to make connections with concepts that they have studied before could aid in the development of adaptive reasoning. Students in Mr. Brown's class had studied similar triangles and the geometric mean earlier in the semester. Their capacity to remember these concepts on their own and, moreover, to justify why they were relevant to the solution of the problem were crucial for developing adaptive reasoning.

FOSTERING A PRODUCTIVE DISPOSITION

To develop strategic competence and the ability to reason adaptively, students must see mathematics as a worthwhile pursuit and develop habits of mind to apply consistent and diligent effort (Kilpatrick, Swafford, and Findell 2001). A key element of a successful problem-based lesson is the chance for

students not only to make sense of the mathematics involved but also to understand the benefits of persevering when the solution is not immediately apparent. In our analysis of the classroom videos, we found evidence of the challenges that students encountered as well as their efforts to overcome those challenges.

One group of three students initially had some difficulty drawing auxiliary lines that would be helpful for finding the length of diameter AB. After looking in their textbook for a relevant theorem, they first found the theorem that states that a line parallel to one side of a triangle divides the other two sides proportionally. Given this information, they chose to construct a line segment from H parallel to $\overline{AB}$ (see **fig. 4**).

An example of students' dialogue as they worked on the problem follows:

Student 1: OK, if a line parallel to the side of the triangle intersects the other two sides, then it divides those two sides proportionally, which means ... But wait, that still doesn't work.

Student 2: Why?

Student 1: Because we don't have a line. Because we don't know what this length is [*pointing to the auxiliary line*]. Because, OK, because then you could say that, um, if they're parallel, then 9.82 or, like, 2.8 over 9.2 equals the diameter over that length. But we don't know what that length is, so then that doesn't work.

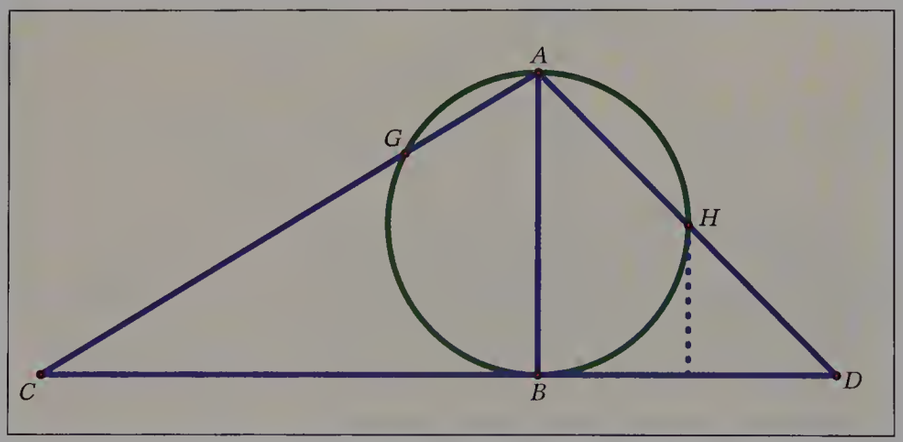


Fig. 4 A first attempt at choosing an auxiliary line was unproductive.

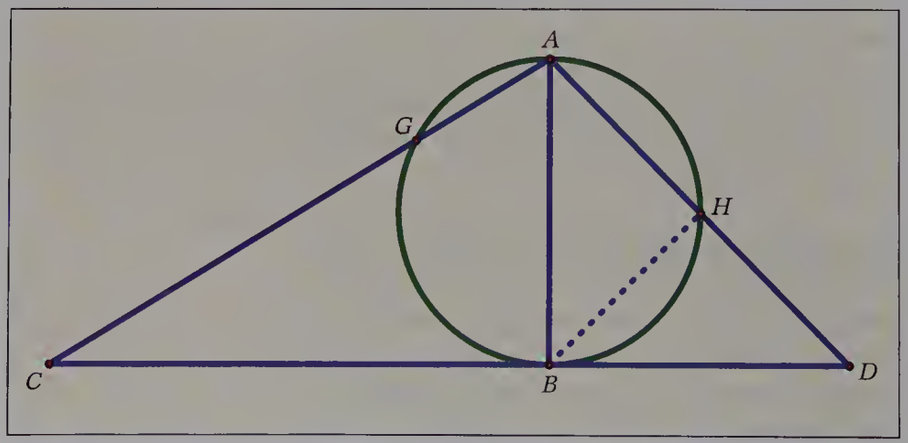


Fig. 5 The group eventually erased its first line and drew $\overline{BH}$.

These students realized that, although they had found a way to draw an auxiliary line and apply the theorem about a line parallel to a side of a triangle, they did not have enough information to make the theorem useful. They erased the line they had drawn and continued to work, eventually arriving at a more successful strategy (see **fig. 5**).

Student 1: Oh, OK, so we can find this right here [*draws* $\overline{BH}$] is equal to this times this [*pointing to* $\overline{AH}$ *and* $\overline{HD}$]. This squared is equal to this times this. So then we can figure out this length [*pointing to* $\overline{BH}$].

These students then used the geometric mean to find BH and applied the Pythagorean theorem to find AB. Although their first attempt did not provide the solution to the problem, it helped them eventually arrive at the correct solution. The students' discussion took into account the need to establish a relationship between $\overline{AB}$ and some auxiliary line, but they learned that certain relationships did not lead to the solution of the problem. Eventually, when they found the relevant theorem and drew the appropriate auxiliary line, they were able to use the information that they had to solve the problem. The students in this group showed a productive disposition because they persisted, learning from previous attempts. Their productive disposition manifested itself in the students' ability to learn from their mistakes by looking for new relationships, studying the diagram carefully, and making connections with theorems that they already knew.

The dialogue excerpts above show that students developed greater mathematical proficiency through the complementary relationship between the different strands. The students showed strategic competence by solving the circle problem. In addition, they showed adaptive reasoning by evaluating different solutions and justifying their solution methods. Their success in solving the problem gave them the opportunity to develop their productive disposition further because they could see how their mathematical efforts paid off.

REASONING AND SENSE MAKING WHEN WORKING ON THE CIRCLE PROBLEM

The circle problem helped Mr. Brown promote reasoning and sense making in his mathematics classroom. We use the four key elements of reasoning and sense making set forth in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009, pp. 9–10) to illustrate how these played out in the problem:

- *Analyzing the problem*—In approaching the problem, students first had to analyze the diagram and find the hidden structure by drawing auxiliary lines.
- *Implementing a strategy*—Students decided on a strategy for finding the length of the diameter and kept track of their computations.
- *Seeking and using connections across different mathematical domains ...* —Students made connections between different mathematical concepts. For example, to apply the Pythagorean theorem to certain triangles, they had to first use the geometric mean or similar triangles to find side lengths.
- *Reflecting on a solution to a problem*—When students implemented an incorrect strategy, this time of reflection allowed them to realize their error and move on to a new strategy. Once they had chosen a more productive strategy, they were able to carry out their computations.

Reasoning and sense making are essential to the development of mathematical proficiency (Kilpatrick, Swafford, and Findell 2001). As a result of our observations of students' work, we propose that the following characteristics of the problem-based lesson built around the circle problem could have supported the reasoning habits that we have discussed and thus allowed students to become more mathematically proficient.

The lesson—

- allowed students to articulate the problem and their strategies for solving it, such as adding auxiliary lines;
- required students to think logically about connections between theorems regarding circles that they had studied recently and theorems about triangles that they had studied in the past;
- required students to take different pieces of knowledge and organize them into a coherent whole—for example, the knowledge about segment lengths that had been provided, the knowledge about characteristics of the diagram that they had to identify, and the knowledge of the theorems and strategies that they could use to solve the problem;
- encouraged students to use procedures efficiently and accurately for identifying unknowns, setting up proportions, and solving equations; and
- gave students the chance to experience the benefits of perseverance by trying different strategies until these strategies were fruitful.

These lesson components can help teachers promote reasoning and sense making in the mathematics class as students work through problems.

Reasoning and sense making are essential to developing the strands of mathematical proficiency. The circle problem required students to reason mathematically, thus enabling them to improve their reasoning as well as their strategic competence and productive disposition. If the goal of instruction is to help students develop mathematical proficiencies, then we should provide examples like the circle problem to give students a chance to develop their ability to reason mathematically.

REFERENCES

Cai, Jinfa, Ning Wang, Bikai Nie, John C. Moyer, and Chuang Wang. 2010. "Student Learning and Classroom Instruction Using Standards-based and Traditional Curricula: An Analysis of Instructional Tasks." Paper presented at the annual meeting of the American Educational Research Association, Denver.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell, eds. 2001. *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academy Press.

Moise, Edwin E., and Floyd L. Downs. 1975. *Geometry.* Reading, MA: Addison-Wesley.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

———. 2009. *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, VA: NCTM.

National Research Council (NRC). 2000. *How People Learn: Brain, Mind, Experience, and School.* Washington, DC: National Academy Press.

Ridgway, James, Judith Zawojewski, Mark Hoover, and Diana Lambdin. 2003. "Student Attainment in the Connected Mathematics Curriculum." In *Standards-Based School Mathematics Curricula: What Are They? What Do Students Learn?* edited by Sharon Senk and Denisse Thompson. Mahwah, NJ: Lawrence Erlbaum.

Ridlon, Candice L. 2009. "Learning Mathematics via a Problem-Centered Approach: A Two-Year Study." *Mathematical Thinking and Learning* 11: 188–225.

GLORIANA GONZÁLEZ, ggonziz@illinois.edu, is an assistant professor of mathematics education in the Department of Curriculum and Instruction at the University of Illinois at Urbana-Champaign. Her research interests include problem-based instruction and the teaching and learning of geometry. ANNA F. DEJARNETTE, fricano1@illinois.edu, is a doctoral student in the College of Education at the University of Illinois at Urbana-Champaign. She is interested in how students make sense of mathematical ideas.

Building Sinus

A measurement-based activity can help students struggling to understand trigonometric functions.

Mara G. Landers

oids

For several decades, NCTM has called for mathematics education to provide students with opportunities to "learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000, Learning Principle, p. 20). This constructivist learning principle applies to students at all levels, but teaching to this principle in the context of already-challenging areas such as trigonometry is a further challenge. This challenge and my desire for students to build knowledge of trigonometric functions in an active way led me to reconsider how I was introducing precalculus students to the sine and cosine functions. In this article, I describe the development and implementation of a measurement-based group activity designed to support students in understanding the connection between angle magnitude and the shape of the sine function.

The definitions and behavior of trigonometric functions differ from functions that students have previously encountered in that "trigonometric functions . . . cannot be expressed as algebraic formulae involving arithmetical procedures, and students have trouble reasoning about such operations and viewing these operations as functions" (Weber 2005, p. 91). When functions are defined with respect to triangles, students must "relate pictures of triangles to numerical relationships, to cope with ratios such as sin A = *opposite/hypotenuse* and to manipulate the symbols involved in such relationships" (Blackett and Tall 1991, p. 144). If the functions are defined on the unit circle, students may lose sight of triangle relationships that justify particular function values. Further, students are challenged to manage and relate two graphical representations: the unit circle and a Cartesian graph.

CONTEXT AND MOTIVATION FOR THE ACTIVITY

When I began teaching at a community college, I found significant variation in students' readiness for precalculus. Previously, when I taught high school mathematics, the majority of my students

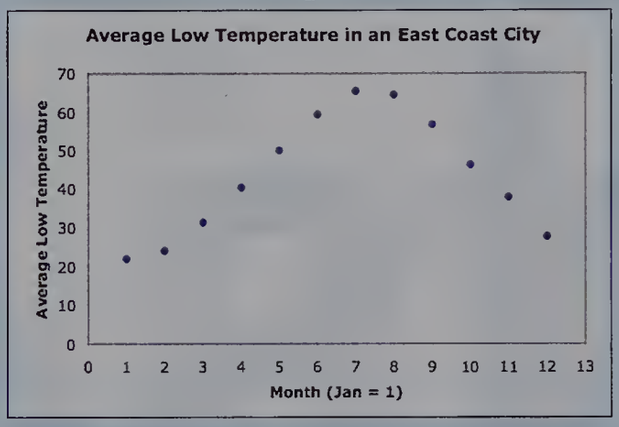
Tasks

Present graph 1 to students with the following questions:

- What would $x = 13$ represent on this graph?
- What would the function value be?
- How do you know?
- How would the graph be different if the data represented where we live?
- How would it be the same?

After students generate comparisons, present graph 2 and introduce the terms *period*, *amplitude*, and *midline*.

Graph 1



Graph 2

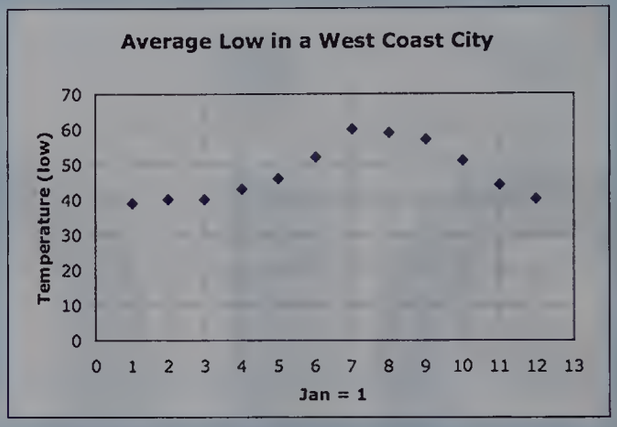


Fig. 1 A weather data task was used to introduce sinusoidal models.

had taken the same second-year algebra course. At the community college, precalculus students have different mathematics backgrounds. Only some have taken the college's developmental courses, which emphasize reasoning, problem solving, and communication. The students' prior knowledge of geometry and of functions, especially trigonometric functions, also varies. Some students are familiar with right-triangle trigonometry, and some have even worked with the unit circle. Other students have no prior experience with trigonometry.

The college's precalculus course focuses on functions and modeling; trigonometric functions constitute a major portion of the course content. Because our students do not take a separate trigonometry course, the trigonometry unit in precalculus is the main—or even sole—opportunity for them to learn about trigonometric functions before calculus. My concern from the start was how to efficiently help students understand these functions mathematically and as models for real-world situations.

For the first two semesters that I taught the course, I introduced sine and cosine as models using periodic data such as weather data (see **fig. 1**). Then we spent several class sessions on angle magnitude, the unit circle, radian measure, and deriving function values of special angles before returning to the graphs of sine and cosine.

After using this sequence of lessons several times, I noticed that students had difficulty understanding the relationship between function values in the unit circle and the shape of the sinusoidal graphs. I attributed these difficulties to the length of time between the introduction of sinusoidal models and our return to graphing. Also, my lesson on parent functions was brief and did not hold students accountable for actively generating graphs. Consequently, I designed the lesson presented here to give students an opportunity to engage actively in building a sinusoid.

DEVELOPMENT AND OVERVIEW OF THE ACTIVITY

The goals of this activity are for students to understand the definition of sine as a function that inputs angles (in standard position) and outputs ratios of vertical distances and to understand the generation of a sinusoidal graph from angles in standard position. Although the shape of sinusoidal functions is often motivated using contexts such as a Ferris wheel (e.g., Connally, Hughes-Hallett, and Gleason 2006; Kaplan 2008), I chose not to contextualize so that students would focus on the definition of the sine function. I also chose not to use triangles (e.g., Petersen, Averbeck, and Baker 1998) so that students would focus their

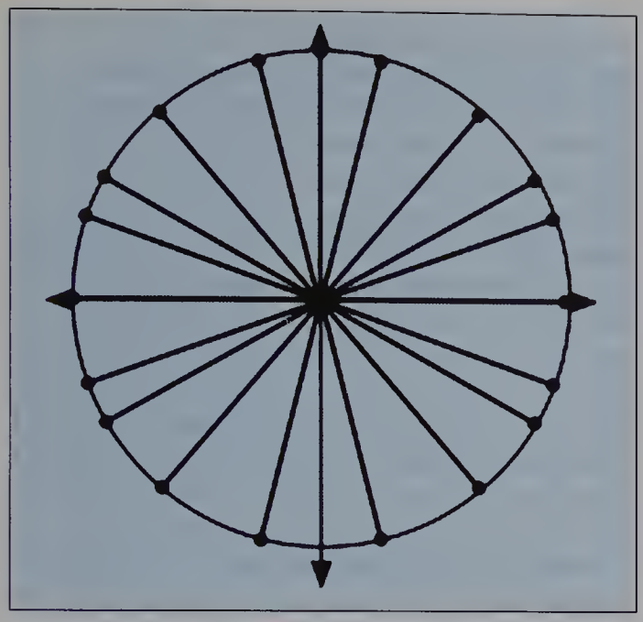


Fig. 2 The circle has angles marked at intervals.

attention on the magnitude of angles, especially angles with measures greater than 360° and less than 0°. (The domain of the sine function should be defined in radians, but the activity uses degrees because of students' prior knowledge. Radians are introduced later in the unit, and then students are expected to define the period as 2π.)

Because I wanted students to generate the sine function by measuring angles and vertical distances, the activity needed to include angles that would generate a representative sample of data points for graphing and circles with radii that could be easily measured. I created a worksheet that shows a circle marked with angles in standard position that are selected multiples of 20°, 30°, 50°, 75°, and 90°. I made five versions of the worksheet, using radii measuring 3 cm, 4 cm, 4.5 cm, 5 cm, and 6 cm, so that each group of students would have a circle with a different radius. A sample of one such circle is shown in **figure 2**. Examples of completed student worksheets are shown in the supplemental material on the Web (**www.nctm.org/mt038**).

The overall plan for the activity is for students to work in groups to complete the following tasks:

- Measure the angles (in degrees) and record these measurements in a table as function inputs.
- Measure the vertical distances (in centimeters) from the x-axis to the point at which the terminal side of the angle intersects the circle; record the distances as function outputs.
- Graph the data to see the shape of a sinusoidal function.
- Create a second function, using the same angles as the inputs but now using the quotients of the vertical distances and the radii as the outputs.
- Compare graphs from all groups to see that all the second functions are the same.

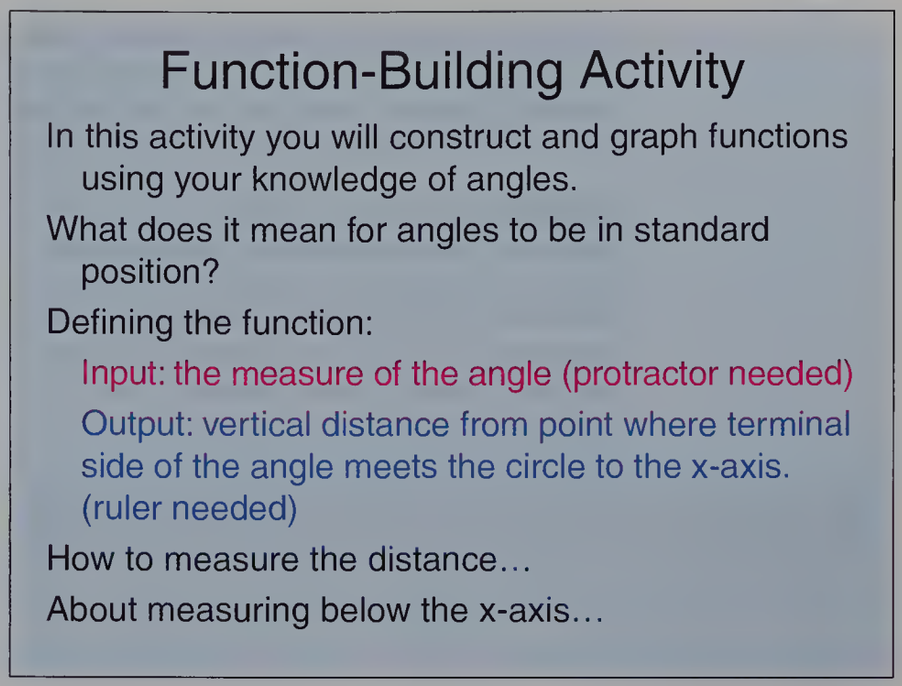

Function-Building Activity
In this activity you will construct and graph functions using your knowledge of angles.
What does it mean for angles to be in standard position?
Defining the function:
Input: the measure of the angle (protractor needed)
Output: vertical distance from point where terminal side of the angle meets the circle to the x-axis. (ruler needed)
How to measure the distance…
About measuring below the x-axis…

Fig. 3 This PowerPoint® slide set the direction of the activity.

The first time I implemented the activity, a colleague observed the lesson and provided feedback about the activity and how students engaged with it. We both found that the choice of angle measures and radii worked well to produce data for students to see the shape of the graph. However, some students struggled to measure the vertical distances. As a result, I revised my plans so that the activity would include a demonstration of how to measure the angles and distances.

We also noticed that some groups abandoned their protractors once they discovered that angle measures could be found using symmetry across an axis. Some groups also used symmetry to make assumptions about the function values rather than continuing to measure distances with a ruler. When I saw how much time students saved by using symmetry, I subsequently encouraged groups to transition from measuring to using symmetry. In addition, having students complete a homework assignment on building a cosine function helps solidify their understanding of how to generate a sinusoidal graph.

IMPLEMENTING THE ACTIVITY

Setting Up

During the class period before this activity, students work on the weather data task (see **fig. 1**), and we define the terms *periodic function*, *amplitude*, and *midline*. In preparation for the activity, I introduce the concepts of an angle in standard position, including negative angles, angles with measure greater than 360°, and co-terminal angles.

To begin, students are given definitions of the inputs and outputs of the functions that they will build (see **fig. 3**). To avoid confusion, before groups

begin to work, I demonstrate how to measure using a document camera. In addition, I suggest that distance be directional, so that the angles with terminal sides below the x-axis have negative function values.

Group Work

Each group receives worksheets, rulers, and protractors. As students work, I circulate around the room to respond to questions and ask some of my own that encourage them to transition from measuring every angle to using symmetry to find the measures of angles beyond the first quadrant. If a group is beginning to measure angles in the second quadrant, I ask the students if they can figure out "how far" an angle is from 90° using the symmetric angle that they have found in the first quadrant. If groups have not discovered on their own that they can then make assumptions about function values using the same symmetry, I ask them to look at their table and identify any patterns they see: "How are the values that you find related to values that you found in the first quadrant? What do you notice about the angles in the third or fourth quadrant?" Generally, students do not find it difficult to complete the tables and graphs, although groups have varied in how quickly they are able to transition from measuring to using symmetry.

When groups complete the table for the first function, they create poster-sized graphs. Then they move on to the second table and graph. When all the graphs are completed, students tape their posters to the wall and take a tour of the room to answer the last question on the worksheet: "Compare your functions to other groups' functions. What do you notice?" See **figure 4** for results for two groups.

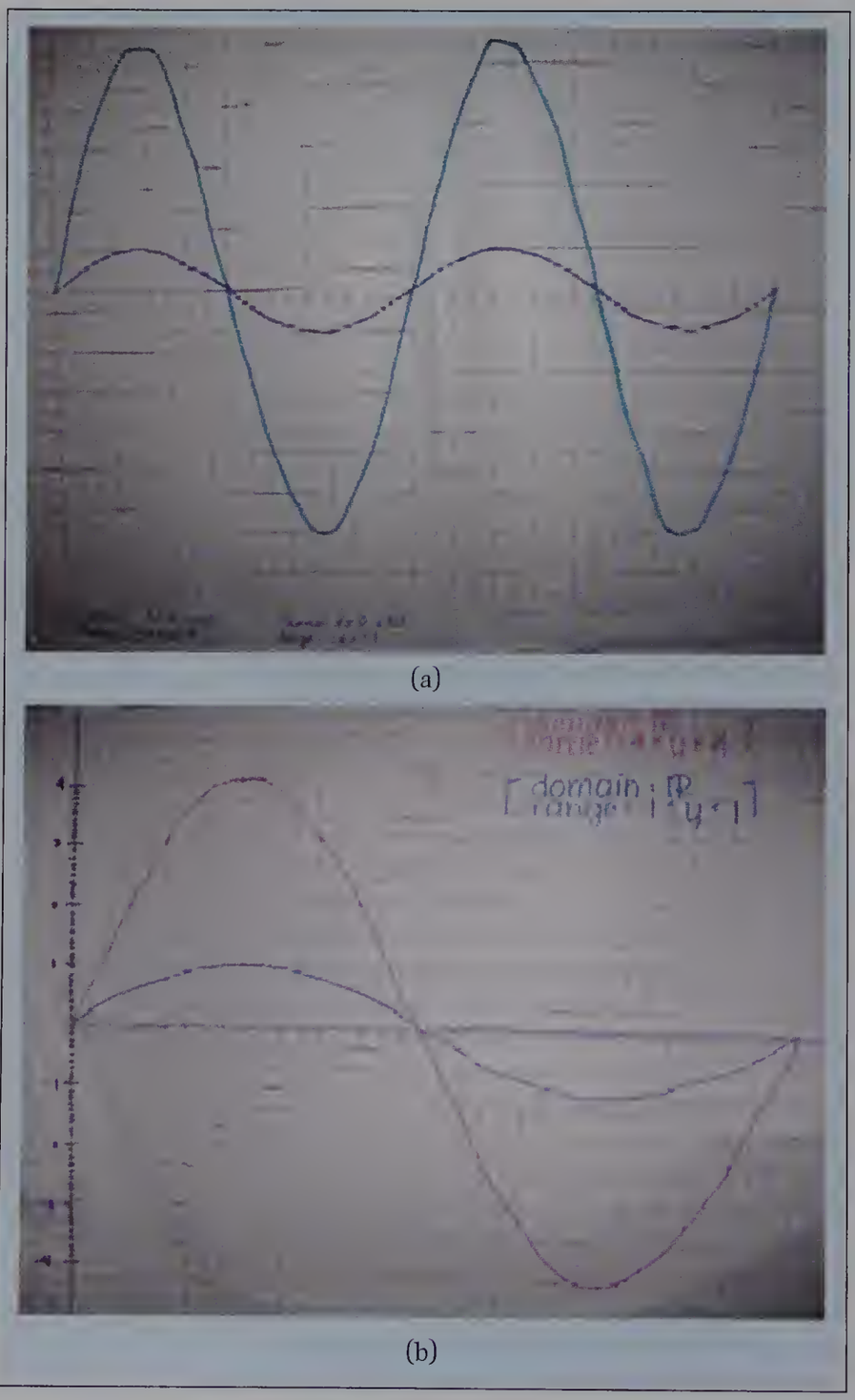


Fig. 4 The student work shown in (**a**) includes a second cycle, but the domain is written in a nonstandard way. The students whose work is shown in (**b**) graphed a single cycle and correctly stated domain and range.

Discussing the Activity

After groups view all the posters, I pose a series of questions to the entire class:

- Find the domain and range of each function.
- How did you extend the functions to create a second period?
- What is the period of each function? How do you know?
- What did you notice about the graphs of the second function?

The first three questions aim to confirm that the groups have correctly interpreted their own graphs and that we agree about the period of the functions. When I have used this activity, all groups were able to identify the period of the functions as 360°, even if they did not extend their graphs to show a second period. The last question serves to confirm that the poster tour leads students to see that the second function is the same for every group. This conclusion leads us to define the sine function as the ratio of the vertical distance to the circle's radius. (The justification that the ratio is unique for a given angle and thus that the sine is a function is discussed below.)

At this point, I introduce a new graphical representation: an angle θ in standard position with $\sin\theta$ defined as the ratio y/r. Students confirm the domain and range of the sine function and also evaluate the accuracy of their measurements by calculating sine values with their calculators.

Using the Function We Defined

After the sine function is defined, I ask students to find sine values of two types of angles in standard position: (1) when a point (x, y) is given (students will need to find the radius) and (2) when the terminal side of the angle coincides with an axis (i.e., the quadrantal angles). In both cases, students conclude that angles can have the same sine values if there is symmetry across the y-axis or if the angles are co-terminal.

Extending the Activity

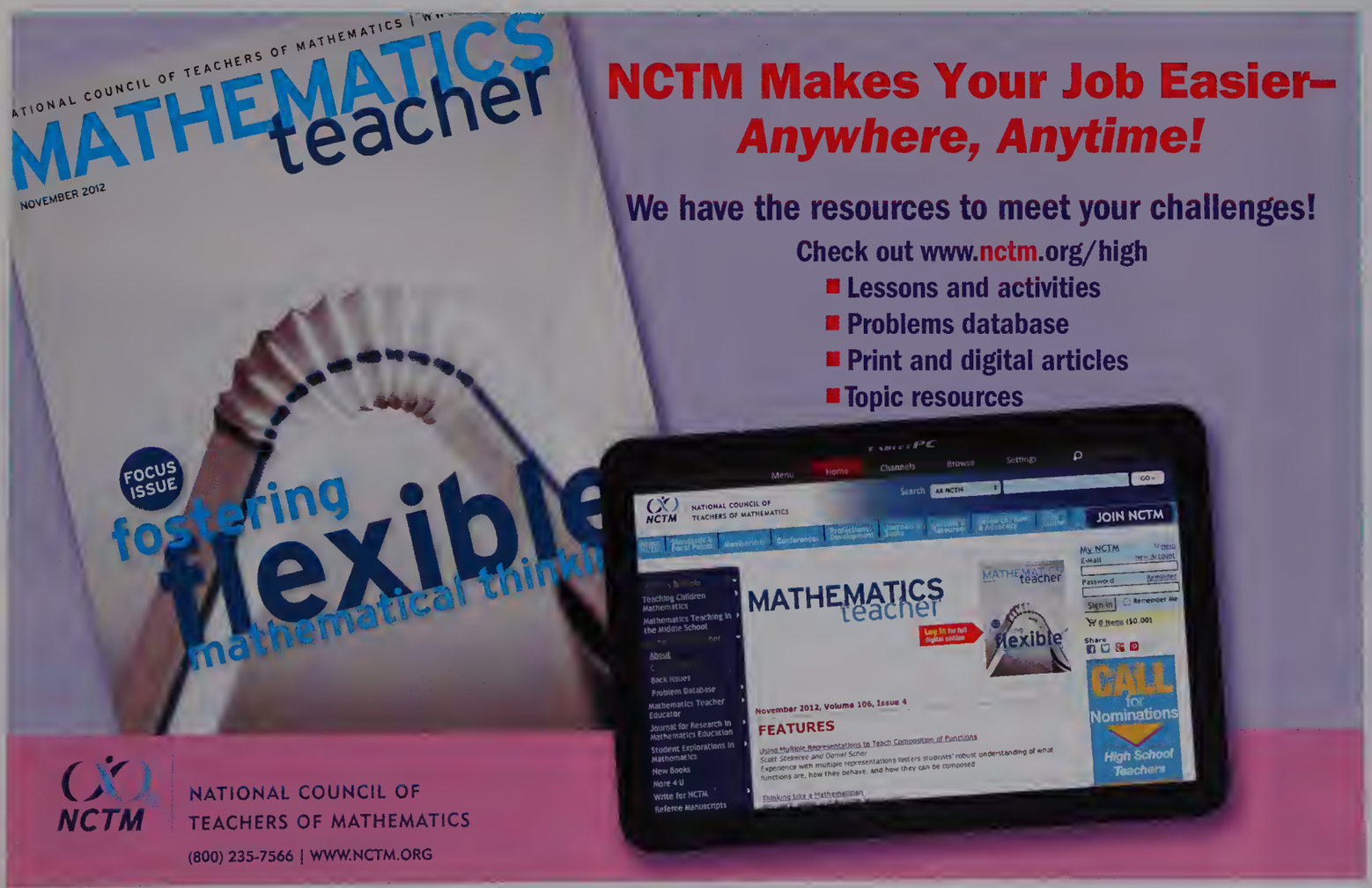
This activity can be extended in several ways. First, students can build the cosine function using horizontal instead of vertical distances. Assigning this task as homework gives students the opportunity to discover the cosine function and to confirm their understanding of how they generated the sine function in class.

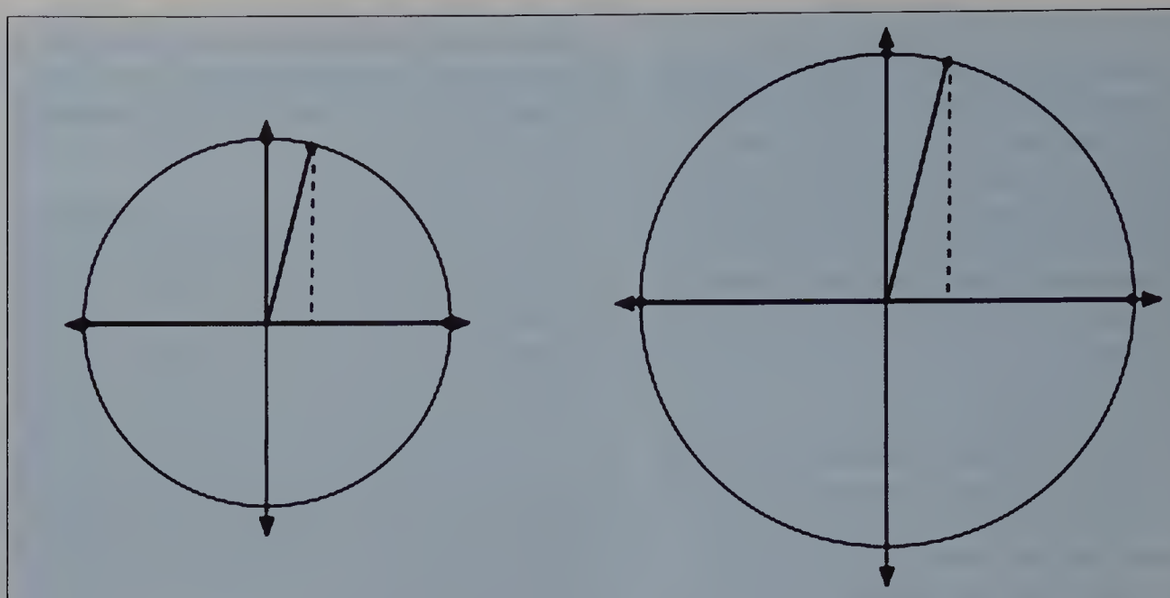
Students should figure out why a given angle on circles with different radii yields the same ratio (functional value). The kind of task shown in **figure 5** can guide students to use similar triangles to justify this idea. However, how teachers engage students with this problem depends on students' prior knowledge and memory of geometry. Students' prior knowledge will be less an issue in a secondary school than it is in my community college context, where nontraditional students may have studied geometry many years ago. In my context, an extension activity requires more class time to review similar triangles.

Finally, students need opportunities to develop the ability to use rotation and symmetry to explain why two angles will have the same sine (or cosine). Providing students with a blank set of axes and asking them first to sketch an angle of approximately 280° and then to sketch a second angle having the same sine enables them to use their circle from the activity as a tool. Because the angle 280° is not part of the activity, students can sketch this angle on their group's circle and revisit the reasoning they used in the activity—that symmetry across the y-axis, or a co-terminal angle, would yield angles with the same sine value.

CONCLUSIONS AND REFLECTIONS

Although I was originally concerned about how much time this activity would take, especially the work of measuring, I have found that students develop a stronger understanding of the relationship between angles in standard position and the shape of the sine and cosine graphs. Having a collection of graphs posted on the walls to revisit





(a) (b)

These images show two circles that groups in your class analyzed previously, with the angle 75° in standard position. The vertical distance from the x-axis to the circle is shown as a dotted line. Use these figures to explain why the second function was the same across all groups. (Hint: How can you prove that triangles are similar?)

Fig. 5 This sample extension task will help solidify student knowledge.

when questions arise is helpful, as is our collective memory of building the sine function. The posters are especially useful when we tackle questions of symmetry, period, and amplitude.

In the future I plan to create another activity of this type—using rotation to investigate the periods of functions of the form $y = \sin bx$ and $y = \cos bx$. Students often resort to memorizing the formula $period = 2\pi/b$ without necessarily understanding how the parameter b influences the period of the sinusoid.

The benefit of this activity is that it allows students to build their trigonometric knowledge actively through a hands-on experience. If we want students to develop deep understanding of functions in general and trigonometric functions in particular, then this type of activity should be an essential part of the curriculum.

REFERENCES

Blackett, Norman, and David O. Tall. 1991. "Gender and the Versatile Learning of Trigonometry Using Computer Software." In *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education*, edited by Fluvia Furinghetti, vol. 1, pp. 144–51. Assisi, Italy: PME.

Connally, Eric, Deborah Hughes-Hallett, and Andrew M. Gleason. 2006. *Functions Modeling Change: A Preparation for Calculus*. Hoboken, NJ: John Wiley and Sons.

Kaplan, Gail. 2008. "Trigonometry through a Ferris Wheel." *Mathematics Teacher* 102 (2): 138–43.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Petersen, Blake E., Patrick Averbeck, and Lynanna Baker. 1998. "Sine Curves and Spaghetti." *Mathematics Teacher* 91 (7): 464–567.

Weber, Keith. 2005. "Students' Understanding of Trigonometric Functions." *Mathematics Education Research Journal* 17 (3): 91–112.

MARA G. LANDERS, mlanders@losmedanos.edu, is an assistant professor of mathematics and co-lead of the Developmental Mathematics Education Program at Los Medanos College in Pittsburg, California. She is interested in reform curricula and teaching practices, particularly those related to homework.

more 4U

For examples of worksheets and student solutions, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt038**.

CALL FOR MANUSCRIPTS

Assessment

Assessment is an integrated part of mathematics instruction that guides and enhances teaching and learning. A key aspect of instructional decision making is the alignment of standards, curriculum, instruction, and assessment. The *MT* Editorial Panel is interested in manuscripts that address one or more of the following themes related to assessment.

Promoting student learning

- What assessment strategies foster opportunities for students to reflect on their own or their peers' mathematical strengths and weaknesses?
- How do students benefit from assessment that highlights mathematical connections?

Respecting and responding to diversity

- What assessment strategies have you tried that honor student diversity?
- How do you negotiate the tension between the diversity of your students' experiences and the goal of being fair?

Driving instructional planning

- How do you use formative and summative assessments in your process of instructional planning?
- How do assessment results guide your next steps?
- Which assessments have been effective and why?

Creating alternative assessments

- What alternative assessments have you developed related to specific mathematical content?
- How do you use new technologies to assess student thinking?
- What did you learn about your students and their mathematical progress from these assessments that you would not otherwise have discovered?

Teaching in a context of external assessments

- How do you balance the content coverage demands of summative assessments with teaching for understanding?
- How do you incorporate "test preparation" into your courses in meaningful ways?

Submit your completed manuscript for review to *Mathematics Teacher* at **mt.msubmit.net**. Be sure to check the box indicating that this manuscript is for the Assessment call. Guidelines for the preparation of manuscripts can be found at nctm.org/publications/content.aspx?id=22602.

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

MAY

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

A chocolate cake baked in a 9 in. × 9 in. square pan is to be divided equally among three people. The three straight cuts will each have one endpoint at the cake's center. If the first cut is made as shown, where should the next two cuts be made?

1

If

$$\sqrt{a}+\sqrt{a}=\sqrt{20},$$

what is the value of a?

2

Stage 1 of a fractal has 3 unit squares that form an L shape. Place three copies of stage 1 in an L shape to create stage 2. Form stage 3 with three copies of stage 2. The perimeters of the three stages (both interior and exterior) are 8, 20, and 56. Create stage 4 and find its perimeter.

3

Without using a calculator, write the following numbers in ascending order:

$$3^{14},\ 4^{9},\ 8^{7},\ 16^{5}$$

4

If

$$f(x)=x-\frac{1}{x},$$

find

$$f(x)+f\left(\frac{1}{x}\right).$$

5

If

$$\sqrt{a}+\sqrt{a}=\sqrt{b},$$
$$\sqrt{b}+\sqrt{b}=\sqrt{c},\text{ and}$$
$$\sqrt{c}+\sqrt{c}=\sqrt{d},$$

write an expression for d in terms of a.

6

A fair coin is tossed 8 times, and the sequence of heads and tails is recorded. What is the probability that the sequence contains at least one pair of consecutive heads or at least one pair of consecutive tails?

7

When 20/13 is written in decimal form, what digit will be found in the 2013th place to the right of the decimal point?

8

Find the value of a if

$$12\log 12=3a\log 3+2a\log 8.$$

9

We have

$$\sqrt{a}+\sqrt{a}=\sqrt{b},$$
$$\sqrt{b}+\sqrt{b}=\sqrt{c},$$
$$\sqrt{c}+\sqrt{c}=\sqrt{d},\ \cdots,$$
$$\sqrt{g}+\sqrt{g}=\sqrt{h}.$$

Find the sum

$$a+b+c+\cdots+h$$

in terms of a.

10

A fraction with a numerator of 1 and a positive integral denominator is called a *unit fraction*. When a fraction is written as a sum of two or more unit fractions, it may be referred to as an *Egyptian fraction*. Write the fraction 13/40 as a sum of two distinct unit fractions.

11

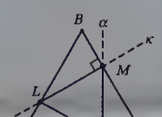
My daughter, my mother, and I were all born on Mother's Day. When my daughter was born, I was 3/7 the age of my mother. Today my daughter is the age I was when she was born; in 4 years she will be exactly 1/3 the age of my mother. How old was I when my daughter was born?

12

The product of three distinct positive integers a, b, and c is 2013^2. What is the largest possible value for the sum $a+b+c$?

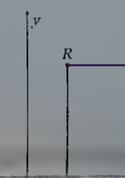
13

In equilateral $\triangle ABC$, point L lies on $\overline{AB}$ such that $AL = LB/2$. Construct a line k through L perpendicular to $\overline{BC}$; let M be the intersection of k and $\overline{BC}$. Construct a line l through M perpendicular to $\overline{AC}$; label the intersection N. Show that $\triangle LMN$ is equilateral and find the fraction of



14

Rectangle $QRST$ lies in quadrant I as shown; its sides are parallel to the x- and y-axes. Q has coordinates (2, 1); R lies on the line $y = -5x + 18$; and S lies on the line $y = 2x/3 + 2$. What are the



15

The integers a and b are such that their arithmetic mean is 6 more than their geometric mean. If $a = 48$, what is the value of b?

16

How many integers n, such that $2 \le n \le 210$, are divisible by neither 3 nor 7?

17

Of the three right triangles with integral sides and perimeter = 120, which has the largest area?

18

Henry drew a convex hendecagon (11-sided polygon) with all its diagonals. Cathy drew two different convex polygons with all their diagonals. She counted the diagonals in each polygon, d_1 and d_2. Henry observed that $d_1 + d_2$ was the same as the sum of the diagonals and the sides in his hendecagon. How many sides did Cathy's two polygons have?

19

An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, how far will it have walked?

20

The coordinates of the seven vertices of a 7-gon are generated by the following rule:

For $x \in \{4, 5, 6, 7, 8, 9, 10\}$,
$f(x) = (x - 7)^2 + 3$.

Find the area of the 7-gon.

21

Find the area of the region bounded by the curve

$$f(x) = x^2 - 14x + 52$$

and the line

$$g(x) = 12.$$

22

Prove that the measures of the four angles formed by the intersection of the altitudes of a parallelogram equal the measures of the four angles of that parallelogram.

23

The lengths of the altitudes of a parallelogram are 20 and 32. They intersect at an angle θ. If $\sin\theta = 4/5$, what is the area of the parallelogram?

24

Three vertices of a rectangle have coordinates (–16, 12), (–6, –2) and (1, 3). Which of the rectangle's four vertices is farthest from the origin?

25

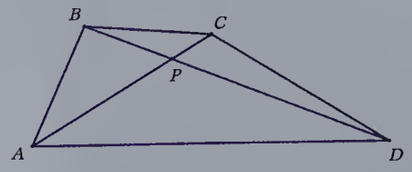
Points A (–5, 0) and B (5, 0) are the endpoints of a diameter of a circle. Point P moves along the semicircular path from B to A. Find an equation that gives the area of $\triangle APB$ in terms of the x-coordinate of point P.

26

Examine the three stages of the fractal shown in the problem for May 3. Create and examine additional stages, if necessary, to develop an explicit formula for the perimeter of the nth-stage fractal.

27

Diagonals AC and BD of quadrilateral $ABCD$ intersect at P. If the area of $\triangle BPC = 4$, the area of $\triangle CPD = 10$, and the area of $\triangle DPA = 35$, what is the area of $\triangle APB$?



28

Points A (–5, 0) and B (5, 0) are the endpoints of a diameter of a circle. Point P moves along the semicircular path from B to A. Find the area of $\triangle APB$ when $m\widehat{AP} = 2m\widehat{PB}$.

29

Since $\sqrt{14 + 6\sqrt{5}} = 3 + \sqrt{5}$, in some sense $14 + 6\sqrt{5}$ is a *perfect square* in the set of numbers of the form $a + b\sqrt{c}$ where a, b, and c, $c > 0$, are integers. Determine the requirements for a, b, and c that make this type of number a perfect square.

30

Consider the following equations:

$$\sin 75° = \sin\left(45 + 30\right)° = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\sin 75° = \sin\left(\frac{150°}{2}\right) = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Using a technique similar to that of the previous problem, show that these answers are unique.

31

© National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502

SOLUTIONS to calendar

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher*.

Edited by ***Margaret Coffey,*** *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and* ***Art Kalish,*** *artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791*

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, $22.95) and the 100 Problem Poster (stock number 13207, $9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

1. 3 in. from a corner (see fig. 1a). A cake is a three-dimensional object; however, if it has uniform thickness, then we need only find two additional cuts that result in equal surface area of the tops of the pieces. Suppose that cuts were made from A to C and from B to C. Then there would be two pieces with surface area $SA = (1/4)(81) = 20.25$ in.^{2} and one piece with surface area $SA = (1/2)(81) = 40.5$ in.^{2}. The area of each of the smaller pieces must be increased by 6.75 in.^{2} to obtain 27 in.^{2}, which is 1/3 of the total area. Find a point P_1 on side BD such that the area of $\triangle P_1CB = 6.75$ in.^{2}. We have $(1/2)bh = (1/2)b(4.5) = 6.75 \rightarrow b = 3$. $P_1B = P_2A = 3$ in.; the second and third cuts will each have an endpoint 3 in. from a corner of the cake (see fig. 1b). Observe that each person will also receive the same amount of frosting since each piece has 12 linear in. of frosting along the outer edge of the cake.

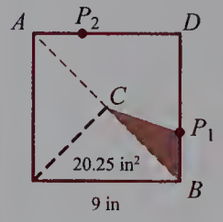


Fig. 1a

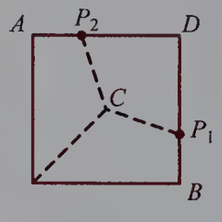


Fig. 1b

2. 5. Since

$$\sqrt{a} + \sqrt{a} = 2\sqrt{a} = \sqrt{4a},$$

$4a = 20$ and $a = 5$.

3. 164. Stage 4 is shown in figure 3a. Figure 3b illustrates what happens when we place three copies of stage 1 together to create stage 2 or when we place three copies of stage 2 together to create stage 3. Four segments of length 1 will become common interior segments. Therefore, the perimeter of stage 4 is 3(*perimeter of stage 3*) − 4 = 3(56) − 4 = 164. The fractal whose construction begins as shown here is a modification of the Sierpinsky triangle. (The building blocks for the figure are squares rather than equilateral triangles.) (For a longer activity for the Sierpinsky triangle, see "Inquiry into Fractals" by Raymond Siegrist in *MT* Oct. 2009 [Activities for Students, vol. 103, no. 3, pp. 206–12].)

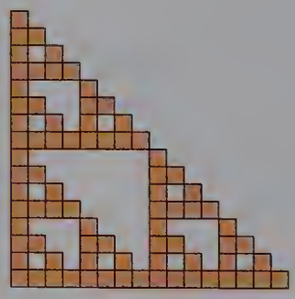


Fig. 3a

Fig. 3b

4. $4^9 < 16^5 < 8^7 < 3^{14}$. Three of the numbers can be written as powers of 2: $4^9 = (2^2)^9 = 2^{18}$; $8^7 = (2^3)^7 = 2^{21}$; and $16^5 = (2^4)^5 = 2^{20}$. Therefore, $4^9 < 16^5 < 8^7$. The number 3^{14} can be written as $(3^2)^7 = 9^7$, so it is the largest number in the list.

5. 0. Consider $f(x) + f(1/x) = x - (1/x) + (1/x) - x = 0$.

6. $d = 64a$. We begin with the result shown in solution 2: $\sqrt{a} + \sqrt{a} = \sqrt{4a}$.

This result implies that $b = 4a$ and also that $c = 4b$ and $d = 4c$. Thus, $d = 4c = 16b = 64a$.

Alternate solution: Since $\sqrt{a} + \sqrt{a} = \sqrt{b}$, we have $2\sqrt{a} = \sqrt{b}$. Square both sides to obtain $4a = b$ and proceed as above.

7. 127/128. If a fair coin is tossed 8 times, there are $2^8 = 256$ possible sequences. If heads and tails strictly alternate, then no outcome will occur twice in a row. There are only 2 ways for heads and tails to strictly alternate: One begins with a head, the other with a tail. The desired probability is $(256 - 2)/256 = 127/128$.

8. 8. The decimal representation of 20/13 can be written as $1.\overline{538461}$. Six digits repeat. Since $2013 = 6 \cdot 335 + 3$, the third digit in the repetend, 8, will be the 2013th digit.

9. 4. Use the properties of logs to rewrite both sides of the equation: $\log 12^{12} = \log 3^{3a} + \log 8^{2a} \rightarrow \log 12^{12} = \log(3^{3a} \cdot 8^{2a})$. Since $8^{2a} = (2^3)^{2a} = (2^2)^{3a}$, we have $12^{12} = 3^{3a} \cdot 4^{3a} = 12^{3a}$. Thus, $a = 4$.

10. $5461a$. Note that h is the eighth letter of the alphabet. Use the result shown in solution 6 to write the sum $a + b + c + \cdots + h$ as $a + 4a + 16a + \cdots + 4^7a$. This is a geometric series with common ratio $r = 4$ and first term a: $S = a(1 - 4^7)/(1 - 4) = (-16383/-3)a = 5461a$.

Alternate solution: Solve for each variable in terms of a as follows: $\sqrt{b} = 2\sqrt{a}$, $\sqrt{c} = 4\sqrt{a} \cdots$, $\sqrt{h} = 64\sqrt{a}$. Square both sides of each equation and add to obtain $b + c + d + \cdots + h = 4a + 16a + 64a + \cdots + 4096a$. Proceed as above or simply add the seven terms and then add $1a$ for the final sum.

11. 1/5 + 1/8. Let $1/a$ and $1/b$ represent the two unit fractions. Then

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{13}{40}.$$

We seek two natural numbers whose sum is 13 and whose product is 40. Those numbers are 8 and 5. We can also solve the following equation:

$$13ab = 40a + 40b$$
$$13ab - 40b = 40a$$
$$b(13a - 40) = 40a$$
$$b = \frac{40a}{13a - 40}$$

Since b must be a natural number, $a > 3$. Using guess and succeed, we try values for a. Before too long, we reach success when $a = 5$, implying that $b = 8$. This result provides a rule for adding any two unit fractions: The numerator is the sum, and the denominator is the product. For example, 1/9 + 1/14 = 23/126. Readers are encouraged to determine three unit fractions whose sum is 13/20 of the denominators.

12. 24 years old. Let m = my mother's age when my daughter was born. Then my age was $(3/7)m$ when my daughter was born. Today my daughter is $(3/7)m$ years old; in 4 years she will be $(3/7)m + 4$. Today my mother is $m + (3/7)m$; in 4 years she will be $(10/7)m + 4$. Set three times my daughter's age in 4 years equal to my mother's age in 4 years:

$$3\left(\frac{3}{7}m + 4\right) = \frac{10}{7}m + 4$$
$$\frac{9}{7}m + 12 = \frac{10}{7}m + 4$$
$$m = 56$$

Finally, $(3/7)56 = 24$.

13. 1,350,727. Choose two factors as small as possible—say, $a = 1$ and $b = 3$. Then the third factor—and the sum of the three factors—will be as large as possible. In this case, the third factor $c = 3 \cdot 11^2 \cdot 61^2 = 1{,}350{,}723$, and the sum $a + b + c = 1{,}350{,}727$.

14. 1/3; the proof follows. Since $AL = LB/2$, let $AL = a$ and $LB = 2a$. We have $m\angle B = 60°$ and $m\angle LMB = 90°$, so $\triangle LMB$ is a 30-60-90° right triangle with shorter leg $MB = a$. Since $BC = AC$, MC must have length $2a$; $\triangle MNC$ is also a 30-60-90° right triangle with shorter leg $NC = a$. Finally, $\triangle NLA$ has $NA = 2a$, $m\angle A = 60°$, and $AL = a$, so $\triangle NLA \cong \triangle LMB \cong \triangle MNC$ by SAS. The long leg of each of these three triangles must have length $a\sqrt{3}$, implying that $\triangle LMN$ is equilateral. All equilateral triangles are similar. Since $AL : LM = 3a : a\sqrt{3} = \sqrt{3} : 1$, the ratio of the areas of the two equilateral triangles is $(\sqrt{3})^2 : 1^2 = 3 : 1$. Thus, $\triangle LMN$ contains 1/3 the area of $\triangle ABC$.

15. (9, 1). Since Q has coordinates (2, 1) and $\overline{QT}$ is parallel to the x-axis, T has y-coordinate 1. Point R must have the same x-coordinate as point Q—namely, 2. Substituting $x = 2$, we have $y = -5(2) + 18 \rightarrow y = 8$. If R has coordinates (2, 8), then S has y-coordinate 8. Substituting $y = 8$, we have $8 = 2x/3 + 2 \rightarrow x = 9$. So T has x-coordinate 9.

16. 12 or 108. We have $(48 + b)/2 - 6 = \sqrt{48b} \rightarrow 48 + b - 12 = 2\sqrt{48b} \rightarrow 36 + b = 8\sqrt{3b} \rightarrow b^2 - 120b + 1296 = 0 \rightarrow b = 12$ or $b = 108$. We confirm that both solutions satisfy the original problem. The arithmetic mean of 12 and 48 is 30, and the geometric mean is $\sqrt{576} = 24$. The arithmetic mean of 48 and 108 is 78, and the geometric mean is $\sqrt{5184} = 72$.

17. 119. There are 209 integers between 2 and 210, inclusive. Of these 209 integers, we have $210/3 = 70$ that are divisible by 3 and $210/7 = 30$ that are divisible by 7. We must avoid double-counting the numbers divisible by both 3 and 7: There are $210/21 = 10$ multiples of 21. Thus, $70 + 30 - 10 = 90$ integers that are divisible by 3 or 7, and $209 - 90 = 119$ integers that are divisible by neither.

18. The triangle with sides of length 30, 40, and 50. The first part of the problem requires us to find three Pythagorean triples with a perimeter (sum) of 120. One approach is to look at the sums of some familiar primitive triples and determine which are factors of 120. We organize our findings in the table below.

Leg a	Leg b	Hypotenuse	Sum (Perimeter)	Factor of 120?
3	4	5	12	Yes
5	12	13	30	Yes
8	15	17	40	Yes

The three "smallest" triples (if we order the triples by the length of the hypotenuse) *all* have perimeters that are factors of 120. Therefore, 30-40-50, 20-48-52, and 24-45-51 are the right triangles whose areas we will compare. Since $20(48) = 960 < 24(45) = 1080 < 30(40) = 1200$, the triangle with sides 30-40-50 has the largest area—namely, 600 units^2.

19. 8 sides, 10 sides. In a polygon with n sides, $n - 3$ diagonals can be drawn from a single vertex. To find the total number of diagonals, divide the product $n(n - 3)$ by 2 to avoid counting each diagonal twice. Henry's 11-gon has $11(11 - 3)/2 = 44$ diagonals and 11 sides for a total of 55 connecting segments. The number of diagonals in polygons with 4 to 12 sides is shown in the table at top right.

If the sum of the diagonals in Cathy's two polygons is 55, then one is an octagon with 20 diagonals, and the other is a decagon with 35 diagonals.

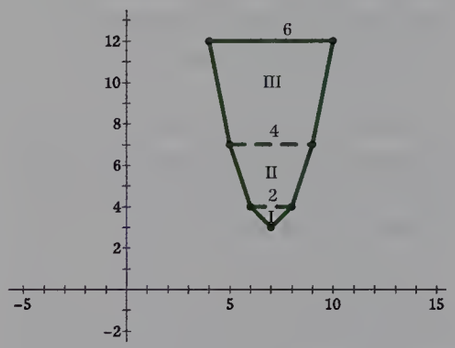
Number of Sides (n)	4	5	6	7	8	9	10	11	12
Number of Diagonals ($n[n - 3]/2$)	2	5	9	14	20	27	35	44	55

20. 36 ft. Since the ant walks the same distance before turning and makes the same 160° turn each time, the turning points must all lie on a circle. The first two segments of its walk can therefore be seen as the sides of an inscribed angle with measure 180° – 160° = 20° (see fig. 20a). The central angle that intercepts the same arc will have measure 40°, implying that there are 360/40 = 9 turning points on the circle, counting the anthill as one of these points (see fig. 20b). The ant travels 9 • 4 = 36 ft.

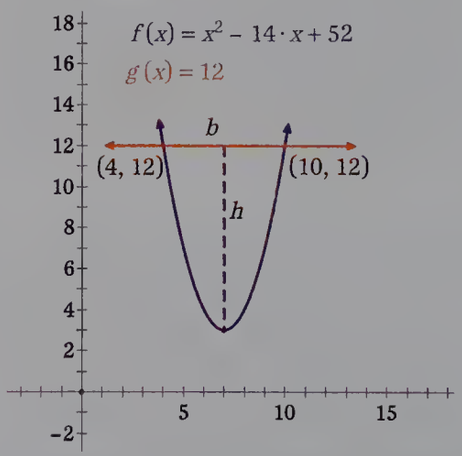
Fig. 20a

Fig. 20b

21. 35 units^2. Plot the ordered pairs (4, 12), (5, 7), (6, 4), (7, 3), (8, 4), (9, 7), and (10, 12). Choose one of several ways to divide the heptagon into regions whose areas are easily computed, exploiting the symmetry of the figure. The figure shows division into two trapezoids and an isosceles triangle. Region I, the triangle, has base 2 and height 1, making the area 1 unit^2. Region II, a trapezoid, has bases 2 and 4 and height 3, making the area 9 units^2. Region III, the larger trapezoid, has bases 4 and 6 and height 5, making the area 25 units^2. The sum of these areas is 35 units^2.



22. 36 units^2. Graph the two functions. Observe that the region bounded by the curve and the line is a parabolic segment. Since $y = 12$ is perpendicular to the parabola's axis of symmetry, use the area formula $A = (2/3)bh$. The graph gives us the values $b = 6$ and $h = 9$, so $A = 36$ square units. (Students who work problems 21 and 22 should realize that problem 21 provides an estimate of the area found in 22. They can extend the problem by finding the percent error of the estimate.)



23. At their intersection, the altitudes of the parallelogram form four angles, whose measures we may designate as a, $180 - a$, a, and $180 - a$. Consecutive angles of a parallelogram are supplementary; designate their measures as b, $180 - b$, b, and $180 - b$. We need only prove that $a = b$ or $a = 180 - b$. In figure 23a, altitudes ZN and XM intersect at P; let $m\angle MPN = a$. Quadrilateral $MPNY$ has two right angles, so $\angle Y$ is supplementary to $\angle MPN$. We have $a + b = 180 \rightarrow a = 180 - b$. We have shown that $m\angle MPN = m\angle WXY = m\angle WZY$; thus, $m\angle XPN = m\angle W = m\angle Y$.

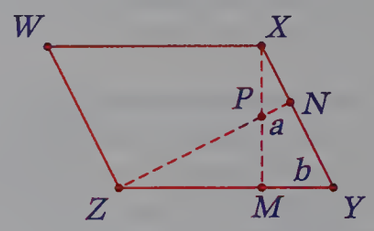


Fig. 23a

Alternate solution 1: Construct the altitude from Y to $\overline{WZ}$; label the foot of the altitude Q. The second altitude measures the distance between $\overline{WX}$ and $\overline{ZY}$; construct this altitude to intersect the first at point Q, as shown in fig. 23b. Let R be the intersection of the altitude with $\overline{ZY}$. Then $\overline{QR}$ is the altitude to the hypotenuse of $\triangle ZQY$. $\overline{QR}$ creates three similar triangles, and $\angle RQY$ with measure a is immediately congruent to $\angle Z$ with measure b, so $a = b$.

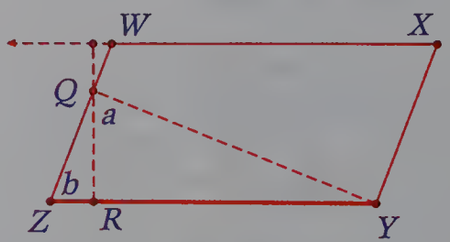


Fig. 23b

Alternate solution 2: Figure 23c is a visual proof.

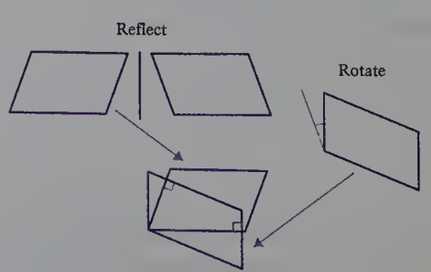
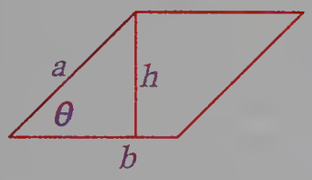
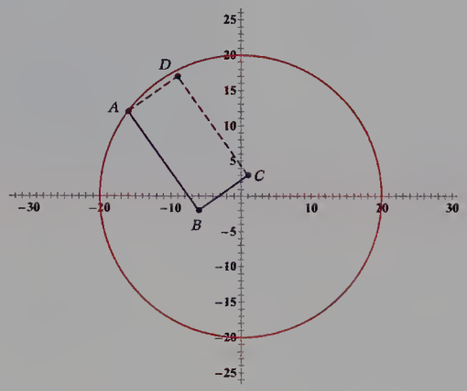


Fig. 23c

24. 800 units^2. Use the result of problem 23 to say that the *sides* of the parallelogram form an angle with measure θ. Let a and b represent the lengths of the sides of the parallelogram, and let h_b and h_a represent the lengths of the altitudes perpendicular to the respective sides. Since $\sin\theta = 4/5$, we have $h_b/a = 4/5$. If h_b is the longer altitude, then a is found as follows: $h_b/a = 4/5 = 32/a \rightarrow a = 40$. The product of a and the corresponding altitude gives the area $A = a \cdot h_a = 40(20) = 800$. If h_b were the shorter altitude, then the value of a is $h_b/a = 4/5 = 20/a \rightarrow a = 25$. The product $a \cdot h_a$ gives the same area: $25(32) = 800$. These calculations demonstrate the following theorem: The area of a parallelogram equals the product of the altitudes divided by the sine of an angle of the parallelogram.

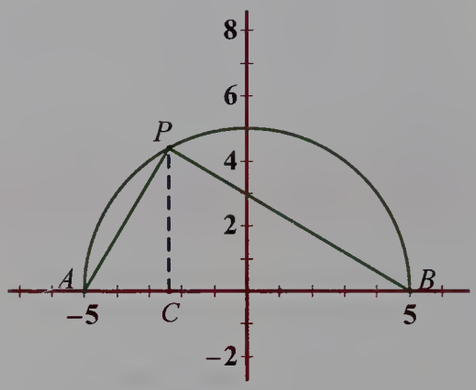


25. (–16, 12). Plot the given vertices to first estimate and then locate the fourth vertex. Using the labels shown in the figure, we observe that point C is 7 units to the right and 5 units above point B in the coordinate plane. The fourth vertex, D, will have the same location in relation to point A. Point D has coordinates (–9, 17). Construct a circle with center at the origin and radius OA; point D lies within the circle.



Alternate solution: Begin as above. After plotting point D, find that the distance from point D to the origin is $\sqrt{(-9)^2 + 17^2} = \sqrt{370}$ and that the distance from point A to the origin is $\sqrt{(16)^2 + 12^2} = \sqrt{400}$.

26. *Area* $= 5\sqrt{25 - x^2}$. A circle with radius 5, centered at the origin, has equation $x^2 + y^2 = 25 \rightarrow y = \sqrt{25 - x^2}$. The value of y gives us the height of the triangle, so $A = 0.5(10)\sqrt{25 - x^2} = 5\sqrt{25 - x^2}$.



Alternate solution: Let C be the foot of the perpendicular segment from P to $\overline{AB}$,

NCTM's Member Referral Program

Let's Add Up! Refer. Receive.

Participating in NCTM's Member Referral Program is **fun, easy, and rewarding.** All you have to do is refer colleagues, prospective teachers, friends, and others for membership. Then as our numbers go up, watch your rewards add up.

Learn more about the program, the gifts, and easy ways to encourage your colleagues to join NCTM at **www.nctm.org/referral.** Help others learn of the many benefits of an NCTM membership—**Get started today!**

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

(800) 235-7566 | WWW.NCTM.ORG

and let x be the x-coordinate of C. Then, $AC = x - (-5) = x + 5$, and $BC = 5 - x$. To find PC, the altitude to the hypotenuse of $\triangle APB$ and therefore the geometric mean between AC and BC, we have

$$\frac{x+5}{PC} = \frac{PC}{5-x} \rightarrow PC^2 = 25 - x^2.$$

The area of $\triangle APB = AB \cdot PC/2 = 5\sqrt{25-x^2}$.

27. If $p_1 = 8$, $p_{n+1} = 3 \cdot p_n - 4$, or for $n \geq 2$,

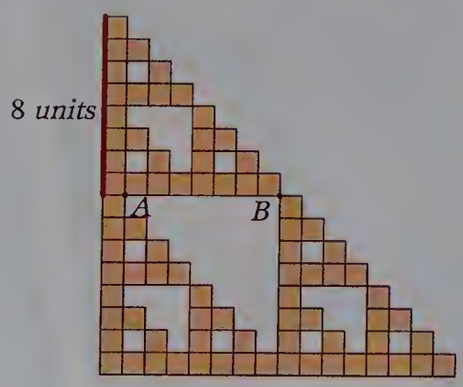
$$p_n = 3^{n-1}(8) - 4\sum_{k=0}^{n-2} 3^k.$$

The perimeter of a stage is 4 less than 3 times the perimeter of the previous stage. Let p_n represent the perimeter of the nth stage. A recursive formula for p_n is $p_{n+1} = 3 \cdot p_n - 4$ where $p_1 = 8$. See also the solution to the May 3 problem.

An explicit formula is a bigger challenge. The sequence of perimeters is not strictly geometric but approaches a geometric sequence with common ratio 3 as the number of stages increases. One strategy might be to organize the "components" of the perimeters of successive stages in a table to observe what patterns develop. In the table at top right, the fourth column shows another way to write the expressions in the third column.

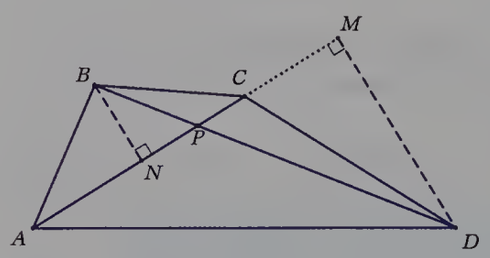
Thus, for $n \geq 2$, an explicit formula for the perimeter is

$$p_n = 3^{n-1}(8) - 4\sum_{k=0}^{n-2} 3^k.$$



28. 14 units^2. Construct the altitude of $\triangle ABC$ from B to $\overline{AC}$. This segment, BN (see the fig. in next col.), serves as a common altitude of triangles ABP and CBP. Therefore, the ratio of their bases, $AP:CP$, equals the ratio of their areas. So the problem of finding the area of $\triangle APB$ is equivalent to finding the ratio of lengths, $AP:CP$. The altitude of $\triangle DCA$ from D to $\overline{AC}$ serves as a common altitude of triangles APD and CPD. Since the ratio of their areas equals the ratio of their bases, we have $AP:CP = 35:10 = 7:2 = 14:4$.

Stage Number	Number of · 8	Number of Perimeter Segments "Lost" in Interior	Equivalent Expression
1	$3^0 \cdot 8 = 8$	0	None
2	$3^1 \cdot 8 = 24$	$1 \cdot 4 = 4$	$3^0 \cdot 4$
3	$3^2 \cdot 8 = 72$	$3 \cdot 4 + 4 = 4 \cdot 4 = 16$	$(3^0 + 3^1) \cdot 4$
4	$3^3 \cdot 8 = 216$	$3 \cdot 16 + 4 = 13 \cdot 4 = 52$	$(3^0 + 3^1 + 3^2) \cdot 4$
5	$3^4 \cdot 8 = 648$	$3 \cdot 52 + 4 = 40 \cdot 4 = 160$	$(3^0 + 3^1 + 3^2 + 3^3) \cdot 4$
n	$3^{n-1} \cdot 8$	?	$4\sum_{k=0}^{n-2} 3^k$



29. $25\sqrt{3}/2$. If $m\overset{\frown}{AP} = 2m\overset{\frown}{PB}$, then $m\overset{\frown}{AP} = 120°$ and $m\overset{\frown}{PB} = 60°$. Since $\triangle APB$ is inscribed in a semicircle, it is a right triangle; its hypotenuse is 10 units. $\triangle APB$ is a 30-60-90° triangle because $m\angle PAB = (1/2)\, m\overset{\frown}{PB} = 30°$. The legs of the triangle have lengths 5 and $5\sqrt{3}$; its area is $25\sqrt{3}/2$ units^2.

30. The number c can be any positive integer, b must be even, and a must equal $(b/2)^2 + c$. These relationships can be discovered by setting $\sqrt{a + b\sqrt{c}} = x + \sqrt{y}$ and then squaring both sides of the equation to get $a + b\sqrt{c} = x^2 + y + 2x\sqrt{y}$. Notice that when $a = x^2 + y$, $b = 2x$, and $c = y$, we have a perfect square.

31. $\dfrac{\sqrt{2}+\sqrt{6}}{4}$. Suppose that $\sqrt{a + b\sqrt{c}} = \sqrt{2 + 1\sqrt{3}}$. Then the technique developed in problem 30 requires that b be an even integer. To make b even, we rewrite $1\sqrt{3}$ as follows:

$$1\sqrt{3} = \frac{2\sqrt{3}}{\sqrt{4}} = 2\sqrt{\frac{3}{4}}$$

So $b = 2$ when $c = 3/4$. Unfortunately, a then is $1 + 3/4 = 7/4$ and not the number 2, as in the problem. Since we want our solution to be in the requested form, we could let $\sqrt{2+\sqrt{3}} = \sqrt{x} + \sqrt{y}$. We square both sides: $2 + \sqrt{3} = x + y + 2\sqrt{xy}$; let $x + y = 2$; and let $2\sqrt{xy} = \sqrt{3} = 2\sqrt{3/4}$, so $xy = 3/4$. Solve this system of equations by substituting $x = 2 - y$ to get $(2 - y)y = 3/4 \rightarrow 4y^2 - 8y + 3 = 0 \rightarrow (2y - 3) \cdot (2y - 1) = 0$. The two solutions for y are $3/2$ and $1/2$; the corresponding values for x are $1/2$ and $3/2$. Using either pair of solutions yields $\sqrt{2+\sqrt{3}} = \sqrt{1/2} + \sqrt{3/2} = (1 + \sqrt{3})\sqrt{2} = (\sqrt{2} + \sqrt{6})/2$. The value for sin 75° is

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

CALL FOR CHAPTERS

Annual Perspectives in Mathematics Education (APME) 2014: *Using Research to Improve Instruction*

INAUGURAL ISSUE

The first volume of APME will focus on using research findings to make a difference for pre-K–16 students' learning by improving instructional practices in classrooms. This issue will help bridge research and practice and will include research-based strategies for equitable and high-level instructional and assessment practices in mathematics classrooms.

Call for Chapters

Using Research to Improve Instruction will focus on bringing research to practice and practice to research in mathematics education. The Editorial Panel encourages chapters that discuss research projects (including classroom-based action-research projects) and collaborations that present findings and information relevant to both researchers and practitioners. In addition to reports of research, the editors welcome manuscripts that discuss research-based practices that influence classroom teaching and learning. The editors also encourage a diverse range of manuscripts that serve to engage readers. Suggested topics include, but are not limited to, the following:

- **Teaching:** Strategies, tools, teachers' instructional moves for effective instruction; effective questioning techniques and other approaches to encourage high levels of student engagement and discourse
- **Learning:** Discussions of children's mathematical thinking and students' learning of different mathematics domains
- **Curriculum:** Selection and implementation of cognitively demanding tasks; teachers' planning and use of curriculum
- **Assessment:** Selection and use of assessment practices for informing planning and instruction and to support students' learning (e.g., formative assessment, progress monitoring, evidence-based assessment, standards-based assessment)
- **Equity:** Equitable teaching practices with attention to diverse learning needs; culturally relevant pedagogy and mathematics for social justice; teaching in rural or urban contexts; multitiered systems of support (e.g., intervention strategies) for at-risk students; encouraging gifted learners; strategies for teaching English language learners
- **Technology:** Use of current and emerging technologies (e.g., dynamic mathematics software, interactive whiteboards, smartphones, smartpens, tablets, online learning environments, apps) in varied instructional settings

Details for Submission

Prospective authors should submit manuscripts for review by **May 15, 2013.** If you are thinking of submitting a chapter, please fill out the **Intention to Submit Form (http://www.nctm.org/publications/content.aspx?id=34981)** by **March 1, 2013,** so that we can follow up with you on your proposed project. Contributions from classroom teachers, school-based mathematics leaders and administrators, teacher educators, mathematicians, and researchers are encouraged.

Editorial decisions will be made by the Editorial Panel: Karen Karp (issue editor), Amy Roth McDuffie (general editor), Barbara Dougherty, Francis (Skip) Fennell, Elham Kazemi, Matt Larson, Travis Olson, Nelson Palmer, and Christine Suurtamm.

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

STUDENTS AS MATHE

If your students learn only one thing in your mathematics class, what would you want that to be? What do you want them to take with them as they walk out your classroom door?

As teachers, we must be able to answer these fundamental questions. Sometimes the answer goes no further than what skills students must master to be successful in their next mathematics class. Sometimes we need to look at the larger context and consider what students need to take with them for the rest of their lives. I try to convince my students that the most important skills that they can take from my classroom are thinking, reasoning, and problem solving. Regardless of their chosen career path and whether or not they think that they will ever need mathematics, students must have these three skills to be successful in life.

To teach students to think, reason, and solve problems, I had to make some changes in my approach to the curriculum. Too often in traditional mathematics classes, concepts are taught in distinct units. Regarding functions, first we teach everything there is to know about linear functions, and then we move on to quadratic functions. Too often, students are left with the mistaken assumption that there is no connection between the two topics. As a result, they do not get to practice recognizing the information in a practical setting, outside a particular unit in their textbook. Consequently, when they encounter a word problem, they assume that it fits the model of the particular function that they are currently studying. Students are not required, then, to be as creative or flexible in their mathematical thought as would be necessary in most real-life settings.

DEBT: ADVENTTR/ISTOCKPHOTO; REGISTER: FOTOFROG/ISTOCKPHOTO;

Five problems—relating to gas mileage, the national debt, store sales, shipping costs, and fish population—require students to use functions to connect mathematics to the real world.

Jennifer L. Jensen

MATICS CONSULTANTS

If students are going to develop reasoning and thinking skills, use their mathematical knowledge, and recognize the relevance of mathematics in their lives, they need to experience mathematics in meaningful ways. Only then will their mathematical skills be transferrable to all other parts of their lives. According to NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* (2009, pp. 9–10), reasoning habits that promote the development of flexible thinking include the ability to apply previous concepts to new problems, the ability to make connections across different contexts and different representations, and the ability to solve problems collaboratively. To promote such flexible mathematical thought, I have my algebra students become mathematics consultants.

THE OBJECTIVE

The goal of this activity is to have students learn how to apply mathematical knowledge in different situations and use this knowledge to make inferences and draw conclusions. I organize students into groups of three, and each group becomes a mathematics consulting firm. The choice of the title "mathematics consultant" is deliberate; it encourages students to interact with the data. Working collaboratively, students are presented with different scenarios and previously collected data. The scenarios represent a broad range of disciplines so that students might begin to see the relevance of mathematics in various situations. However, the real-life application of these scenarios is limited because the issues inherent in the scenarios depend on more factors than those investigated in the problems. The

value lies in students having the opportunity to choose the appropriate function to model the data.

The learning outcomes of the project include identifying different types of functions; analyzing real-world data; identifying dependent and independent variables; determining which type of function would most closely model the situation; using technology, including spreadsheets and graphing calculators, to create graphs and find best-fitting equations; using the equations to make predictions and inferences; and using technology to present conclusions to the class. Students are required to use multiple representations of functions, including verbal, graphical, algebraic, and tabular representations. These objectives align directly with those in the Algebra Standard for grades 9–12 set forth in NCTM's *Principles and Standards for School Mathematics* (2000, p. 296).

PROCEDURE

Before students form the mathematics consulting firms, I give them a short assessment to test their knowledge of functions, concentrating on linear, quadratic, exponential, and cubic functions. This simple tool asks them to work in two directions: to identify a graph by its equation and to graph a function given its equation. The same test is used as a posttest after students complete the project.

The first time I used this exercise only three of twenty-one students could name two or more types of functions. The only function that students could identify by sight or draw a picture of was a linear function. Only two students could name an application of functions. Not one student demonstrated mastery (over 80%) with regard to knowledge of functions. These results surprised me because we had studied different types of functions throughout the year. However, these results confirm my belief that students struggle to recognize mathematical concepts outside the unit in which these concepts are taught.

Then, as a class, we discuss the different types of functions—linear, quadratic, cubic, quartic, exponential, and logarithmic—that we have studied throughout the year. Although students do not always recall these functions at first, with prompting they can usually demonstrate some knowledge. After the discussion, I give them a sample data set that we analyze together, and we review how to use both the graphing calculator and a spreadsheet program, such as Excel®, to input the data and generate the best-fit equation.

After this thorough review, students meet in their groups and are given scenarios from five different "clients" who have hired their firm to analyze and interpret a specific set of data, draw conclusions, and make recommendations. Each scenario represents a different type of function. After students complete the different scenarios, each group chooses one situation to present to the class, using PowerPoint®, a format that they might one day use to make a presentation to a client.

A SYNOPSIS OF THE SCENARIOS

Scenario 1: Gas Mileage Legislation

A local congressman must decide whether or not to vote for a bill to reduce the speed limit, the argument being that gas mileage improves at lower speeds. Students are given a chart of the gas mileage of a particular car at different speeds (see **fig. 1a**). When they plot the data on the computer or calculator, they discover that the function that most closely models the data is a quadratic function (**fig. 1b** shows a sample graph generated by one group of students). Students must use the equation to determine the optimum speed for cars to travel, make a prediction about gas consumption at higher speeds, and make a recommendation to the congressman about how to vote on this legislation. The data for this scenario come from NCTM's Illuminations lesson "Paying for Your Wheels" (Marquez).

Scenario 2: National Debt

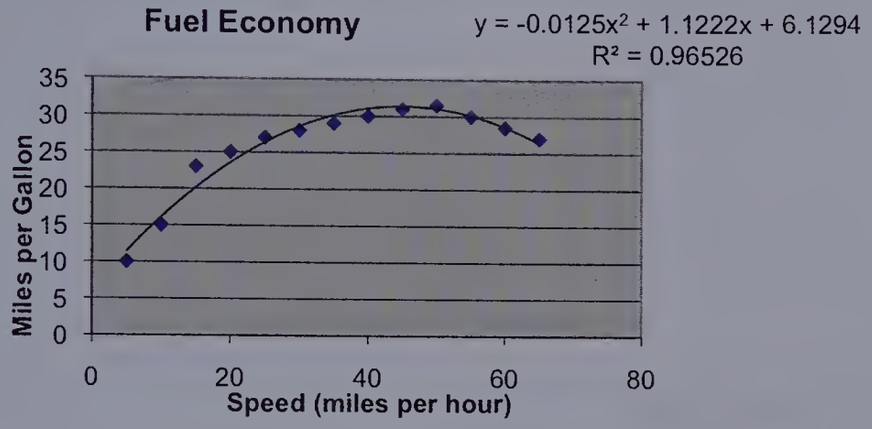
A U.S. senator seeks a mathematical analysis regarding the national debt before she can make a decision regarding a vote on an economic bailout package. Students must research the national debt for each decade, beginning in 1791, found on the Bureau of Public Debt's website (http://www.publicdebt.treas.gov/); see **table 1** for a sample of these data. When students enter the data in Excel, they determine that an exponential function most

Table 1 National Debt

Year	Amount (in millions of dollars)
1791	75
1801	83
1821	90
1841	5
1861	91
1881	2,069
1901	2,143
1921	23,977
1941	48,961
1961	28,8971
1981	997,855
2001	5,674,178

Speed	5	10	15	20	25	30	35	40	45	50	55	60	65
Mpg	10	15	23	25	27	28	29	30	31	31.5	30	28.5	27

(a)



(b)

Fig. 1 The data shown in (**a**) can be used to obtain a graph and an equation (**b**) that will help determine the speed that maximizes fuel economy.

closely models the data. Students must calculate the current national debt based on their equation and compare it with the actual national debt found on the website. Then students make inferences about the difference in these two numbers and make a recommendation to the senator regarding her vote.

Scenario 3: Clothing Store Sales

A clothing store manager wants help analyzing sales for the year so that he can make decisions regarding staffing and inventory. Using the total sales for each month, students create a spreadsheet and graph the data. See **figure 2a** for the data set provided to the students (Stewart, Redlin, and Watson 2004). A quartic function appears to be the best model for these data (see **fig. 2b**). However, some students find that a fifth- or sixth-degree polynomial provides a better fit. This discovery also allows for discussion of whether a higher-degree polynomial function makes logical sense in this scenario. After determining the function that most closely models the data, students draw conclusions about the store's profitability at different times of year. They also make recommendations to the store owner regarding staffing and inventory.

Scenario 4: Shipping Costs

The Civil Aeronautics Board wants to analyze shipping costs. Students are given data regarding the cost of shipping freight a certain distance. When they graph the data, they discover that a linear function most closely models this information. Once they determine the equation, they make predictions regarding the cost to ship crates varying distances. In addition, on the basis of the derived equation, students make recommendations to their client regarding the necessary budget for shipping.

Scenario 5: Fish Population Study

In the final scenario, students analyze data collected by the National Oceanic and Atmospheric Administration (NOAA) regarding the length and age of a certain species of fish (Stewart, Redlin, and Watson 2004). The details of this scenario are based on my own experience aboard a NOAA vessel conducting a study on Pacific hake. Some background on this study, the methods of data collection, and the data given to the students are provided in **figure 3**. After analyzing the data, students determine that the data are best modeled by a cubic function, and they make predictions regarding the age and length of the fish. Then they make recommendations to NOAA about their findings.

SOME FINDINGS

Students spend about seven days in the computer lab working with their group. As I monitor them during the project, they debate which function best models each scenario. As they deliberate and reconcile their differences, they exhibit flexible mathematical thinking. The groups are able to come to their decisions correctly without my prompting, although occasionally they ask me if they have drawn the right conclusion. The most questions

arise on the first day of the project, and these usually relate to how to use the calculator or computer. As the project progresses, however, students gain proficiency in both calculator and computer skills.

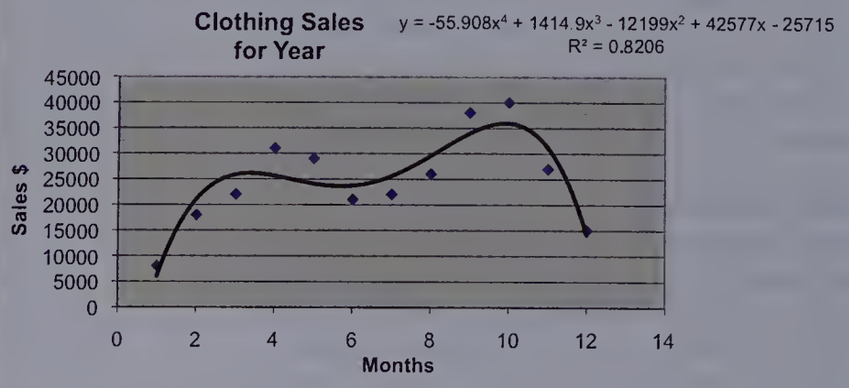
The first time I used this project, each group correctly identified the function that most closely modeled the given data. The groups also identified the dependent and independent variables correctly 80 percent of the time, although the graphs were correct 100 percent of the time. Making inferences and recommendations proved more difficult for students, but they were able to make accurate predictions based on the equations generated. After the first administration of the posttest, all twenty-one students were able to name the five functions that were presented in the different scenarios, with five students naming additional functions. Further, all twenty-one students could define the concept of a function and identify applications of functions, including some not in the scenarios.

At the end of the project, students complete a short survey. They often describe their experience as "fun," "useful," and "interesting." One student wrote, "It was interesting to learn about the different problems in the world." Another student indicated that he liked working "on stuff that is relevant." Overall, the students' responses to the project are positive, and they demonstrate improved knowledge of functions, their applications, and evidence of flexible mathematical thought.

Total Sales for 1 Year

Month	Sales (in dollars)
January	8,000
February	18,000
March	22,000
April	31,000
May	29,000
June	21,000
July	22,000
August	26,000
September	38,000
October	40,000
November	27,000
December	15,000

(a)



(b)

Fig. 2 Clothing sales based on the data shown in (**a**) can be modeled with a quartic equation (**b**).

PROJECT BENEFITS

Students must understand that when they encounter mathematics in the real world, there will be no flashing neon sign indicating the appropriate function. They need to be able to make connections among different graphical representations, equations, and real-world data. And they need to be able to think and reason, both critically and creatively.

This activity provides students the opportunity to develop all these skills. I plan to add to my repertoire of scenarios to include other types of functions or other fields of study. This practice will enrich the experience for students and further promote flexibility of mathematical thought through teaching the mathematical standards.

REFERENCES

Bureau of Public Debt, Department of the Treasury. http://www.publicdebt.treas.gov/.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———. 2009. *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, VA: NCTM.

Marquez, Elizabeth. "Paying for Your Wheels." NCTM Illuminations. http://illuminations.nctm.org/LessonDetail.aspx?id=L708.

Stewart, James, Lothar Redlin, and Saleem Watson. 2004. *College Algebra*. 4th ed. Belmont: Brooks/Cole-Thomson Learning.

JENNIFER L. JENSEN, jjensen@warren.k12.in.us, teaches mathematics at Warren Central High School in Indianapolis while she pursues a PhD in curriculum and instruction (mathematics education) at Indiana State University in Terre Haute. Her interests are promoting critical thinking and curriculum development.

SCENARIO 5: FISH POPULATION STUDY

Client 5: The National Oceanic and Atmospheric Administration (NOAA)

The National Oceanic and Atmospheric Administration (NOAA) is a government organization that studies the ocean and the atmosphere. Every three to five years, it commissions a population study of Pacific hake, a type of fish that lives off the coasts of Washington, Oregon, and California. The study attempts to determine the health of the population and set fishing limits. It is important that the oceans not be overfished.

Analyzing the health of the hake population includes documenting the size and age of the fish. Obviously, each hake in the ocean cannot be counted, so a sample is taken. For some fish in this sample, NOAA scientists remove an ear bone, the *otolith*. The otolith can help determine the age of the fish because each year's growth results in a new ring added to the bone; these rings are similar to growth rings in trees. After removing this bone, scientists record the length of the fish it came from. These data can then be used to draw conclusions about the age and size of a fish.

During its most recent hake study, NOAA recorded the data below. NOAA has hired your mathematics consulting firm to analyze the data. Your job is to determine what type of function might best model these data. Then you will need to use this model to answer some important questions for your client.

Age (yr.)	Length (in.)	Age (yr.)	Length (in.)
1	4.8	9	18.2
2	8.8	9	17.1
2	8.0	10	18.8
3	7.9	10	19.5
4	11.9	11	18.9
5	14.4	12	21.7
6	14.1	12	21.9
6	15.8	13	23.8
7	15.6	14	26.9
8	17.8	14	25.1

Instructions

1. Using Excel, input the data into two columns and create a scatter plot. Don't forget to title the graph and label the axes.

2. Looking at the graph, determine what kind of function appears to represent the data best. _____

   _____

   Add the trend line to your graph. Don't forget to display the equation and the r^2 value on your chart. Are you sure that you have the best equation? Try a few different equations to make sure that you get the best fit.

3. What is the independent variable? _____

4. What is the dependent variable? _____

5. What is the equation? _____

6. What is the value of r^2? _____ What does this value tell you? _____

   _____

7. How well does this function fit the data? _____

   _____

8. Now enter the data into a graphing calculator. Use the instruction sheet if necessary.

   What window should you use?

9. X-max: _____ X-min: _____
   Y-max: _____ Y-min: _____

10. Suppose that a fisherman catches a fish 20 in. long. How old is the fish? (Use a graphing calculator to help you answer this question.) _____

11. Do you think that these fish will keep getting larger their entire life? Why or why not?

    _____

12. How big do you think the fish should be before they are caught to eat? Why?

    _____

    _____

13. How could this information help scientists make decisions? _____

    _____

    You may want to use information from the Internet to help you answer some of these questions.

Fig. 3 The author's experience aboard a NOAA vessel provides a personal, real-world backdrop for scenario 5.

A
B

Is That Square Really a Circle?

Considering circles in taxicab geometry helps students with Euclidean concepts.

Christopher E. Smith

When you think of a circle, what is the first image that comes to your mind? Some people might visualize a perfectly round, two-dimensional shape. Others might conjure up some defining characteristics of a circle, such as the radius or the diameter. Still others might recall some more memorable or interesting facts about a circle—for example, that the perpendicular bisector of a chord is a centerline—and perhaps an image of a proof of that theorem.

While teaching a methods class for preservice secondary school mathematics teachers, I found that although all could draw a reasonably close approximation of a circle, not all could provide an entirely accurate definition of a circle. Most students gave a description rather than a definition. For example, most students characterized a circle as being "a round shape," but a parabola is curved as well, so that description was not specific enough. Students added the condition that a circle is a closed shape, but then we needed to distinguish between ellipses and circles, which are both closed, smooth curves. Some students even began tracing a circle in the air with a finger to demonstrate the shape, leading to a brief comparison of examples and definitions.

4X6/ISTOCKPHOTO

We eventually narrowed down our definition to be sufficiently specific, and I abbreviated it to "a locus of points equidistant from a given point." The term *locus* was not familiar to all the students, and even after we defined the term, this new, shorter definition was clearly still causing some conceptual problems.

This discussion led me to think about ways of reintroducing my students to the idea of a locus and the notion of a circle as a set of points rather than—or perhaps in addition to—the idea of a circle as a single object or image. When we talk about circles, we make a usually reasonable assumption that we are looking at a Euclidean version of a circle. This realization inspired me to introduce my students to the non-Euclidean geometry system known as taxicab geometry and to explore some familiar shapes, although in forms that would make them question how they think about these objects as well as their definitions. In addition to giving students pedagogical insights into the topics of distance and conic sections, I wanted to challenge them to understand a familiar topic from a different perspective, thus strengthening their own conceptualization of these important ideas.

GETTING INSIDE THE TAXICAB

Even though all students had experience with Euclidean geometry, very few had experience working with non-Euclidean geometries. The first step, then, was to define for them what we mean by taxicab geometry and what the differences are between this geometry and the one with which they were familiar. I used *Taxicab Geometry: An Adventure in Non-Euclidean Geometry* (Krause 1987) as a source of inspiration and examples.

I started by giving students a diagram with point A at (0, 0), point B at (0, 3), and point C at (2, 2). When I asked which point—B or C—was closer to A, all the students soon discovered the answer by using the Euclidean distance formula. They found that the distance from A to C was $\sqrt{8}$ and that the distance from A to B was 3; then, since $\sqrt{8}$ is less than $3 = \sqrt{9}$, they correctly concluded that C must be closer to A than B is.

Then I changed the rules and the scenario slightly: The lattice points were street corners, and students needed to take a taxicab from corner A to either corner B or corner C. The distance would then be the number of city blocks covered during this taxicab trip along the most direct routes. The intuitive sense that B is still closer to A than C is was still accurate, but the means of measuring these distances was now different and required a different approach. The distance from A to B is still 3, but now the cab ride from A to C will be 4 blocks, some combination of 2 blocks east and 2 blocks north.

When working with taxicab geometry, students often have to sort out a few assumptions, and they did so in this problem. Although the idea of taxicabs and buildings gives the problem a charming physical context, we can, with some examination and discussion, extend this situation from its naturally discrete sense to a more continuous case. Removing the buildings but still maintaining the restriction that the taxicab can drive only parallel to the x- or y-axis allows it now to drive any real number of blocks. This created a greater sense of continuity and allowed us to draw line segments with greater conceptual confidence. We also needed to establish that the taxicab distance is a possible shortest cab ride from one point to the other and concede that in some cases this path might not be unique.

Our next problem was more in line with what was to follow:

> Alice and Bob live at A (1, 3) and B (5, 1), respectively. They want to open a business together as partners, but they do not want the location to be closer to one than the other, so they choose a corner that is an equally distant cab ride to each house. Find the locus of possible store locations.

Some students found the midpoint (3, 2) using the Euclidean method and verified that it satisfied the given taxicab ride condition. However, if they continued with the Euclidean method, their locus for the given problem would be incorrect, because the locus would look like that shown in **figure 1a**. After giving students some time to work on the problem, I wrote the following hint on the board just below the problem: "Corner (4, 4) is a valid location for their store, but corner (5, 6) is not."

An intermediate step that some students took was to find points (2, 1) or (4, 3) or both in addition to their midpoint of (3, 2). They then followed the advice that any two points define a line and constructed the line shown in **figure 1a**. In the Euclidean version of this problem, this approach would be perfectly valid, but here it leads to a faulty solution. After working on this problem for a few minutes in small groups, most students came to the correct taxicab geometry solution (see **fig. 1b**). Again, most had a hard time accepting that locus as the solution because it was not a straight line. This example showed students how to avoid making assumptions about the shapes of the loci that we would be looking for and to be cautious about making generalizations based on a small amount of information. Drawing something as simple as a line, unless it is fully reasoned and justified, could convert the beginnings of an accurate solution into an incorrect one.

THE CIRCLE, REVISITED

Our next problem was to look at a circle, although I deliberately did not use the word in the problem statement:

> Alice is allowed to walk only 4 blocks from her home at A (1, 2). However, Alice likes to push her boundaries. She wants to find all the locations that are as far away as possible from her home. Find this locus.

All the students found the points N (1, 6), E (5, 2), S (1, –2), and W (–3, 2) easily (see **fig. 2a**). Many students recognized that this was a problem about a circle (because we had been talking about circles so much lately) and that these points would also fit if we were considering a Euclidean circle; thus, it did not seem to be too different from the equidistant locus problem. However, after working on the problem, students discovered that the shape of the locus resembled a square (see **fig. 2b**). They all agreed that this problem defined a circle, but the very different shape that emerged took them by surprise.

This cognitive dissonance was a motivator for discussing this problem not as a circle problem but as a locus problem, especially because all taxicab geometry circles take this shape. Aware of the assumptions that students made while working on the perpendicular bisector problem, I wanted them to begin with as few preconceived notions as possible.

This problem is easily extended into another problem with an unexpected solution:

> Bob lives in a house at B (4, 5). Being a little younger than Alice, he is allowed to walk only 2 blocks away from home. Are there any corners where Alice and Bob can meet? If so, how many?

Students recognized that we were looking at the intersection of two circles with distinct centers, so some expected the solution to be no points, 1 point, or 2 points, all of which would be correct if we were working in Euclidean geometry. Students were surprised to see that the two circles intersected to form a line segment (see **fig. 3**).

ALICE'S RESTAURANT AND BEYOND

I wanted students to consider loci in taxicab geometry for the other conics: the parabola, ellipse, and hyperbola. The next problem brought out some interesting observations about students' understanding of parabolas:

> Alice lives on corner M (1, 4) and decides that she wants to open a fish restaurant with fish from the river that runs along the line $y = -2$ (see **fig. 4**). She wants her restaurant to be the same distance from her house as it is from the river where she will have to get the fish. Find the locus of possible restaurant locations. (Assume that the distance to the river is measured as the minimal distance to the river.)

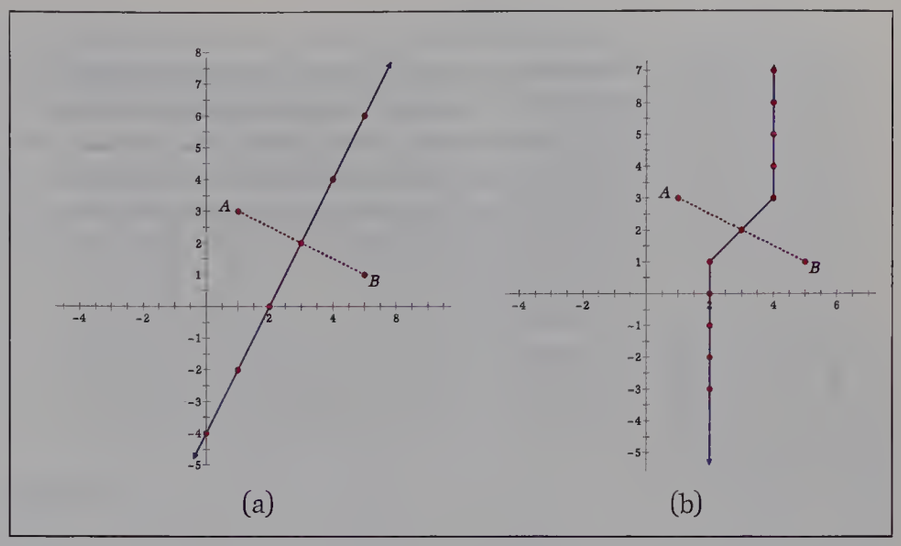


Fig. 1 The Euclidean perpendicular bisector (**a**) differs from the taxicab perpendicular bisector (**b**).

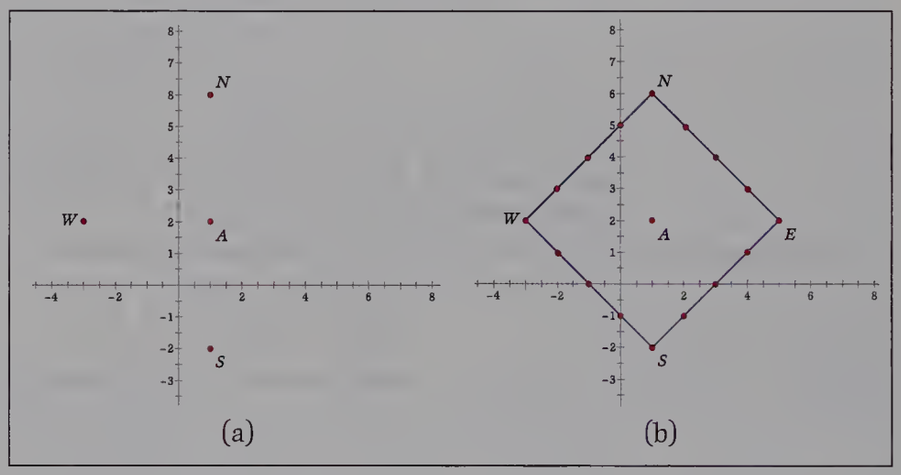


Fig. 2 The first four points on the circle were easy to determine (**a**). A circle in taxicab geometry has a shape that provoked discussion (**b**).

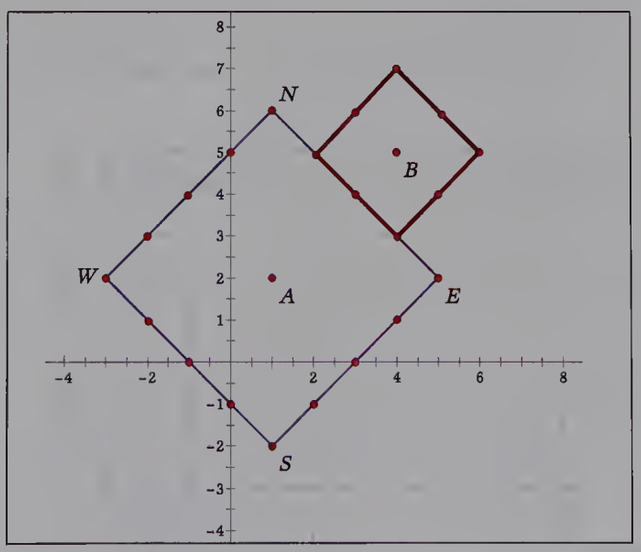


Fig. 3 Two taxicab circles with different radii intersect at a line segment.

Students found the point V (1, 1) fairly quickly. However, even students who recognized that this shape was a parabola seemed to struggle a little after this when they realized that the taxicab locus would be very different from their expectations. Some students noticed that there was no other point besides (1, 1) that was also a taxicab distance of 3 from Alice's house. So they drew a line at $y = 2$ and looked for points on this line that were a taxicab distance of 4 from Alice's house as a possible location for her restaurant. After some inspection, they found the points (–1, 2) and (3, 2). Following the same technique for a taxicab distance of 5 from the river, they found the points (–3, 3) and (5, 3). Then, however, some students reverted to the same mistaken assumption that they had made with the locus of equidistant points problem—that the figure would continue in the V-shape that it appeared to be taking (see **fig. 5a**).

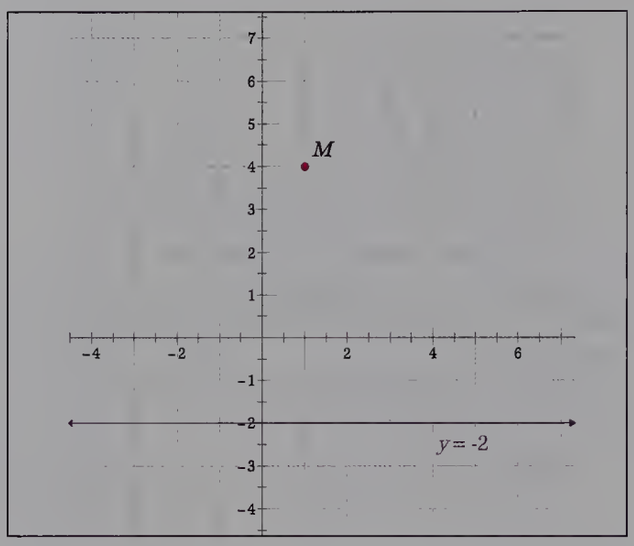


Fig. 4 The parabola problem setup puts the river along the line with equation $y = -2$.

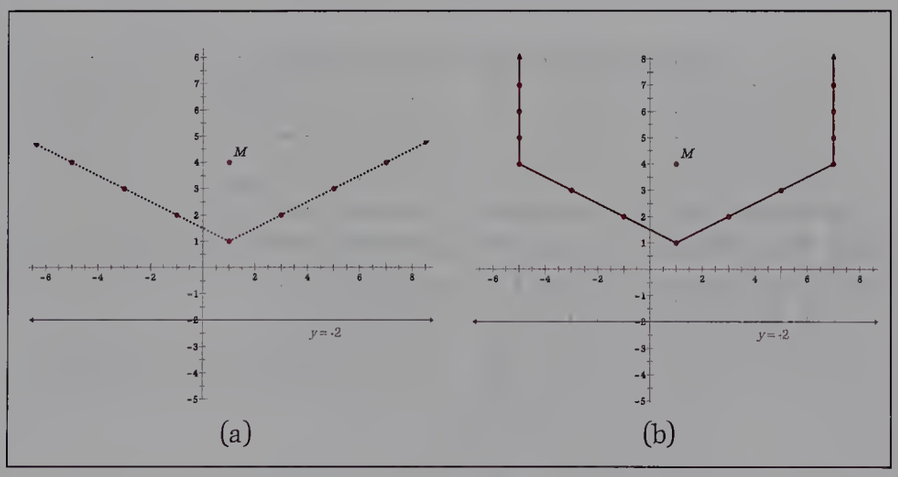


Fig. 5 The V-shaped solution to the parabola problem (**a**) only approximates the correct solution (**b**).

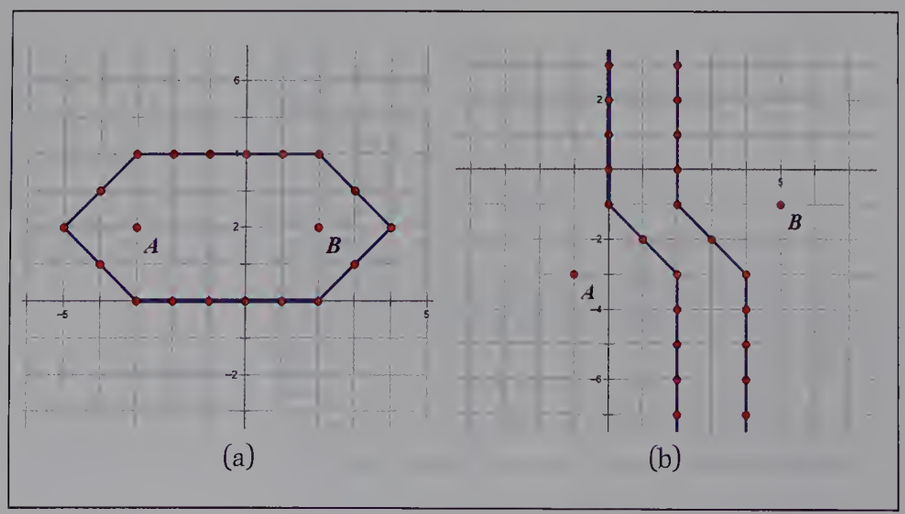


Fig. 6 The graph of the ellipse (**a**) and the graph of the hyperbola (**b**) look only vaguely like their Euclidean counterparts.

After seeing this mistake several times, I gave students the following hint: "Corner (5, 3) is a solution point, but corner (9, 6) is not." This hint reminded them to continue looking for points in the locus even when they felt confident about what the shape of the locus was. The solution (see **fig. 5b**) was another surprise for them: Even though it resembled a parabola, it had significant differences: It was not a smooth curve that was differentiable at every point, and it had a restricted domain.

Even after students had discussed the problem, graphing a parabola without an equation was clearly a difficult task for them. I asked students to point out the focus and vertex of the parabola, but the most interesting part of the discussion concerned the importance of the river in the problem, especially because many admitted that they were no longer familiar with the term *directrix*. Defining this term and others still had a lot of conceptual utility, especially for students who would be teaching this topic soon. The ability to recite an equation or have an inert mental picture of a specific example is far from truly understanding objects as common and important as circles and parabolas.

The final two problems for the students follow:

> Alice and Bob live in houses at A (–3, 2) and B (1, 2), respectively. They are allowed to meet only if they travel a total combined distance of 8 blocks from their houses. Find the locus of possible meeting points.
>
> Alice and Bob live in houses at A (–1, –3) and B (5, –1), respectively, and want to open a store. The absolute value of the difference between the distances between their houses and the store is always 2 blocks. In other words, to get to the store, one of them must travel 2 blocks farther than the other. Find the locus of possible store locations.

By this time, the entire class was able to figure out that these two problems involved the ellipse and the hyperbola, respectively.

The ellipse was the easier of the two to graph but still required a fair amount of care (see **fig. 6a**). Most students found the vertices on the major axis first and then found the minor axis vertices; they completed the ellipse by finding the points between adjacent vertices and assuming symmetry, checking those points to make sure that they were, in fact, part of the locus. The hyperbola was much more complicated because if the major axis is not tilted away from horizontal, the graph consists of just two vertical lines. Giving the two foci different y-values allowed us to see two things about this taxicab hyperbola: (1) it has no asymptotes; and (2) it has only rotational symmetry around the center and no reflective symmetry (see **fig. 6b**).

WHERE THE TAXICAB TOOK US

Students in this class raised several interesting questions:

- Were the reflective properties of conics still satisfied?
- How could these conics be extended into other non-Euclidean metrics?
- If these shapes are conics, what would the cone look like? Are these four figures actual sections resulting from cutting that cone with a plane, as in Euclidean space?

We also discussed whether we could connect the lattice points—if the cab could not drive through the blocks, how could we draw line segments through the blocks? Because students were now comfortable enough with the idea of taxicab geometry, I gave them the alternative physical condition that the cab could drive only in the four main compass headings. Another alternative was that the streets as well as the blocks in between were of negligible width, allowing nonlattice points to be generated. The taxicabs would still have to obey the four main directions of the given condition. These issues provoked discussion and were good indicators that students were giving significant thought to this new system.

Students came away from these problems with a deeper and richer understanding of conics and the idea of a locus. There is nothing wrong with knowing the equations that generate the conics or having a mental image of examples; however, understanding should not be limited to just these symbolic representations. This lesson was used with college students, but the motivation for teaching it and the ideas presented are certainly applicable to classrooms with younger students as well, either as an introduction to forthcoming concepts or as a review before transitioning from a more synthetic approach to a more analytic approach.

I vividly remember reading a lecture by Ludwig Wittgenstein in which he presented a diagram of a circle and a straight line that did not appear to intersect. He commented that we should recognize that "the line cuts the circle but in imaginary points" (Wittgenstein 1939, 1976). This statement made me realize that my own assumptions can prevent me from seeing a problem in its full depth. I saw a lot of the same excitement in my students, especially at moments when they were realizing that something as strange as a square circle, under the right conditions, can exist.

ACKNOWLEDGMENTS

The author would like to thank Timothy Peil of the University of Minnesota at Moorhead and Terry Wyberg of the University of Minnesota for their assistance in developing this article.

REFERENCES

Krause, Eugene F. 1987. *Taxicab Geometry: An Adventure in Non-Euclidean Geometry.* New York: Dover Publications.

Wittgenstein, Ludwig. (1939) 1976. "Lecture 1." In *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge*, edited by Cora Diamond, p. 16. Chicago: University of Chicago Press.

Editor's note: Two NCTM sources will provide more background on taxicab geometry from different perspectives: "Triangle Construction in Taxicab Geometry," by Keith M. Dreiling in *MT* February 2012 (Delving Deeper, vol. 105, no. 6, pp. 474–78); and "Hey, Taxi," by Peggy House, *Student Explorations in Mathematics,* February 2005 (http://www.nctm.org/publications/article.aspx?id=22334).

CHRISTOPHER E. SMITH, smithc2@augsberg.edu, is assistant professor of secondary mathematics education at Augsburg College in Minneapolis. His research interests are in combinatorial reasoning and conceptualizations.

Elemen
Con

GRAFFITI: LABSAS/ISTOCKPHOTO;
CALLIGRAPHY: AKIRASTOCK/ISTOCKPHOTO

ary Algebra • nections to Precalculus

Roberto López-Boada and
Sandra Argüelles Daire

Students use elementary algebra concepts to solve trigonometric and logarithmic equations and systems.

During a typical first-year algebra course, mathematics students learn to solve many different types of equations. They first master linear equations and then move up to the realm of the quadratics. In their second year of algebra, students extend their equation-solving repertoire quite a bit, and by the time they reach courses such as precalculus, they feel that they can solve any equation or system of equations that is presented to them. Often they are right.

As a result of this level of comfort and expertise and with the goal of enticing precalculus students to search for and use connections among mathematical topics, we ask them to solve equations or systems specifically designed to allow simple and elegant solutions to seemingly complex problems. In all the examples that follow, the trick is to find the basic mathematical idea that can be used to simplify the task. Initially, many students use what we call the "graffiti" method, which, although correct, usually creates a much more complex problem than that initially posed. With practice and experience, however, students surprise us as well as themselves by using what we tell them is "calligraphy."

As defined by Webster's online dictionary, *graffiti* is "a rude decoration inscribed on rocks or walls," and *calligraphy* is "the art of elegant handwriting." Our use of these terms in mathematics classes refers to the differences between equally correct but distinguishable techniques in problem solving; both require work and training but of a different kind. We tell students that they have used the graffiti method when they have arrived at the correct answer or solution to a problem by performing unnecessary steps and sometimes by overcomplicating their thinking process.

SOLVING THE SYSTEM USING THE GRAFFITI METHOD

Consider the following system of equations:

$$5 \log_{\frac{1}{7}} x - 3 \log_7 y = -16$$
$$7 \log_7 x + 3 \log_7 y = 20$$

At first, students are distracted by the fact that the bases are not all 7. We enjoy the ensuing class discussion about how to proceed. After some prompting, students recall that

$$\log_{\frac{1}{b}} M = -\log_b M$$

and proceed to rewrite the system as follows:

$$-5 \log_7 x - 3 \log_7 y = -16$$
$$7 \log_7 x + 3 \log_7 y = 20$$

At this point, many students apply the laws of logarithms and rewrite the system:

$$\log_7 \frac{x^{-5}}{y^3} = -16$$
$$\log_7 \left(x^7\right)\left(y^3\right) = 20$$

Eliminating the logarithm by writing the equations using exponents presents students with yet another set of equations:

$$\frac{x^{-5}}{y^3} = 7^{-16}$$
$$\left(x^7\right)\left(y^3\right) = 7^{20}$$

Through substitution and some algebraic manipulation, students eventually obtain $x = 7^2 = 49$.

SOLVING THE SYSTEM USING THE CALLIGRAPHY METHOD

A more elegant solution starts at step 2 from above:

$$5\log_{\frac{1}{7}} x - 3\log_7 y = -16$$
$$7\log_7 x + 3\log_7 y = 20$$

Adding these two equations causes the y term to drop out, yielding $2\log_7 x = 4 \rightarrow \log_7 x = 2 \rightarrow x = 49$.

Substitution into the second equation yields $7\log_7 49 + 3\log_7 y = 20 \rightarrow 7(2) + 3\log_7 y = 20 \rightarrow \log_7 y = 2 \rightarrow y = 49$. When students complete this problem using the calligraphy method, they realize that the process of elimination that they mastered in elementary algebra can also be used to solve nonlinear systems.

The following system can also serve to show students that using previously acquired skills is essential in their progress in mathematics and can definitely simplify their work:

$$\begin{array}{l} x - y + 1 = 0 \\ 4^y + 3 \cdot 2^x = 1 \end{array} \rightarrow \begin{array}{l} y = x + 1 \\ 4^y + 3 \cdot 2^x = 1 \end{array}$$

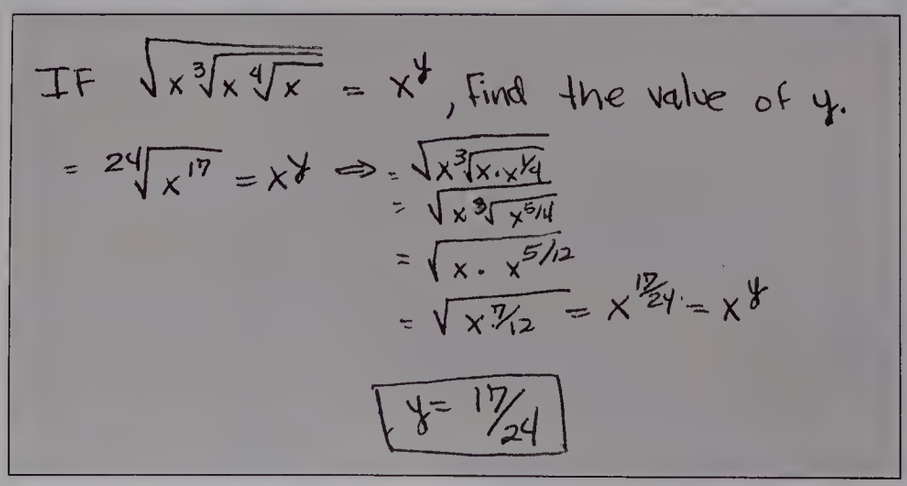

IF $\sqrt{x\sqrt[3]{x\sqrt[4]{x}}} = x^y$, find the value of y.
$= \sqrt[24]{x^{17}} = x^y \Rightarrow = \sqrt{x\sqrt[3]{x \cdot x^{1/4}}}$
$= \sqrt{x\sqrt[3]{x^{5/4}}}$
$= \sqrt{x \cdot x^{5/12}}$
$= \sqrt{x^{17/12}} = x^{17/24} = x^y$
$y = 17/24$

Fig. 1 Despite a careless copying error, this student obtained the correct answer.

We can now rewrite $4^y + 3 \cdot 2^x = 1$ as $2^{2y} + 3 \cdot 2^x = 1$ or $2^{2(x+1)} + 3 \cdot 2^x = 1$. If we then substitute z for 2^x, we obtain a factorable quadratic in z with roots -1 or $1/4$. Discarding the negative value for $z = 2^x$, we obtain the solution $(-2, 1)$.

This type of problem solving (which we want students to do naturally in higher-level mathematics classes) is not the norm in textbooks or in many classrooms. We have found that most precalculus equations—whether logarithmic, rational, exponential, or even trigonometric—involve only one variable. Integrating previously learned algebraic skills and concepts while considering precalculus ideas allows students to achieve a deeper level of mathematical understanding.

Being aware of and exploiting connections among mathematical topics—that is, using calligraphy methods—can result in elegant solutions to higher-level equations and systems. Students' solutions to the problems that follow are further examples.

The following problem, which appeared as a *Mathematics Teacher* Calendar problem (*MT* Sept. 2009, vol. 103, no. 2, p. 125), is not a two-variable system like those previously described. However, it can also challenge students to try a unique approach:

> If
> $x^y = \sqrt{x \cdot \sqrt[3]{x^4\sqrt{x}}}$,
> find the value of y.

A student's solution is shown in **figure 1**.

Another student tried to work with the two sides of the equation simultaneously but got frustrated when the work became cumbersome. Once we discussed the solution above in class, however, many students realized that flexibility is a key factor in this type of problem solving.

To help students build this flexible thinking process, we sometimes present problems that follow such sequential logic. For example, we asked a student to solve the following linear system and explain her thinking process:

$$x + y = 2$$
$$x - y = 2$$

We then asked the student to solve the following three simple trigonometric equations:

$$\sin x = 0$$
$$\cos y = 1$$
$$\sin x = 1 \text{ where } 0 < x < \pi$$

She was then presented with the problem shown in **figure 2**.

The student did not explore the combinations that give a solution $n + l = 1 + 1$, $1 - 1$, and $0 + 0$. However, by applying previously learned mathematics in a new context, she was able to devise a simple yet elegant solution to the system. When presented with a new problem, she was able to apply a similar method and solve it without any guidance as to what substitutions to make.

MAKING CONNECTIONS

We should make clear that we do not frown on students getting a correct answer through the graffiti method; rather, we invite them to seek unique, simpler, and more graceful solutions. We seek elegance in mathematics and make sure that by the time our students enter calculus, they recognize it. In addition, we concur with the observation that "one of the main reasons that students have trouble with calculus is that their algebra is weak" (Kahn 2010, p. xiii). Thus, we use every opportunity we can to weave old and new concepts together, always encouraging students to make connections between elementary algebra and higher-level mathematics.

Our goal in constructing and searching for problems like those presented here has always been to help students make connections among mathematical ideas. The National Board for Professional Teaching Standards (NBPTS) emphasizes this idea in their mathematics standards for both Adolescence and Young Adulthood (AYA) and Early Adolescence (EA): "Accomplished mathematics teachers understand significant connections among mathematical ideas and the application of those ideas in mathematics, in other disciplines, and in the world outside school" (Standard II/EA Mathematics, NBPTS, p. 29). Moreover, "they recognize that mastering mathematical facts and procedures is only part of what it means to learn mathematics. Teachers must understand and consistently employ mathematical thinking processes in their classroom practice that include ... connecting ideas, concepts, and representations across the strands of mathematics" (Standard VI/EA Mathematics, NBPTS, p. 56).

As the problems were presented, we realized that what prevents many top students from being great problem solvers is that they lack flexibility in their thinking. They also seem to lack awareness of their metacognitive processes as they solve the given tasks. In reviewing the NBPTS Standards once again, we found that "in the classrooms of accomplished teachers, students ... search for connections and solve problems while reflecting on both the mathematics and their own thought process" (Standard VI/AYA Mathematics, p. 55).

As we reflect on our own teaching practices, we seek ways to address our own students' lack of awareness of their thought processes, in the hope of incorporating them into our classroom routines. We want our mathematics students to grow in their ability to think outside the box and be aware that they are doing it.

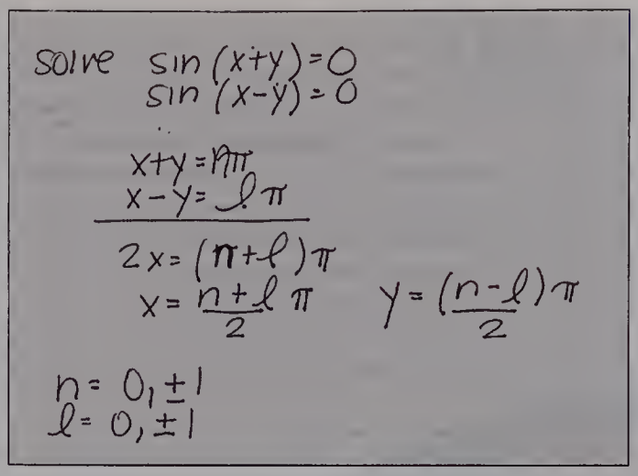


Fig. 2 This student's solution is concise and elegant.

BIBLIOGRAPHY

"Connections." http://www.nctm.org/standards/content.aspx?id=3176.

Denisheva, L. O., Ryazanovsky, A. R., Semenov, P. V., and Sergeey, I. N. 2009. *Federal Bank of Questions: Collection of Mathematical Problems.* Moscow.

Kahn, David S. 2010. *Cracking the AP Calculus AB and BC Exams.* New York: The Princeton Review.

Lidski, V. B. y Otros. 1978. *Problemas de Matemáticas Elementales.* 2nd ed. Editorial Mir.

National Board for Professional Teaching Standards (NBPTS). Mathematics/Adolescence and Young Adulthood. http://www.nbpts.org/the_standards/standards_by_cert?ID=3&x=60&y=8.

———. Mathematics/Early Adolescence. http://www.nbpts.org/the_standards/standards_by_cert?ID=8&x=58&y=11.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Webster's Online Dictionary. http://www.websters-online-dictionary.org/.

ROBERTO LÓPEZ-BOADA, rlboada@dadeschools.net, teaches mathematics at Felix Varela Senior High School in Miami. In his spare time he studies the mathematics of the stock market. SANDRA ARGÜELLES DAIRE, sdaire@dadeschools.net, is a National Board Certified Teacher working on her doctorate in instructional technology and distance education. After twenty-one years of teaching high school mathematics, she has transferred to a K–8 center.

Brandon R. Hanson

Podcasting Potential for High School Mathematics

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

Edited by ***Larry Ottman***
lottman@gfsnet.org
Germantown Friends School
Philadelphia, PA

James Kett
j.gkett@gmail.com
Singapore-American High School (retired)
Singapore

By the end of 2002, about one year after the introduction of the iPod®, Apple estimated that around 600,000 units had been sold (Apple Inc. 2011). As of September 1, 2010, that number had skyrocketed to 275 million, an increase of more than 45,000 percent. This number, taken in consideration with the popularity of the iPhone® and the iPad®, is a sign that the digital age has arrived for most of the country, especially its youth.

Podcasting represents one way in which educators can use these new technologies to facilitate the learning process. A *podcast* is an audio or video file that can be sent directly to these or other capable devices immediately after being uploaded. The files can be downloaded manually, but often an online aggregator, such as iTunes®, is used to automate this process. For educators, this means that they can create files related to lessons and transmit them directly to the students. Unfortunately, little research has focused on the uses of podcasting in grades K–12 (Hew 2009). Noticeably absent as well are studies that focus on the benefits of podcasting in mathematics education. This article highlights four reasons for using podcasting based on current research and provides examples of podcasts that exemplify these reasons. Also provided are tips for creating and distributing podcasts and considerations of potential benefits and pitfalls for podcasting in a high school mathematics classroom.

THEASIS/ISTOCKPHOTO

TIP 1:
You Need Proper Hardware

For recording audio, you will need a built-in or an external microphone that can be connected to a computer. For video, you will need a recording device such as a webcam or a digital camera and a tripod. If you plan to record your work on a whiteboard or chalkboard, consider using a high definition digital camera.

REASON 1: PODCASTS CAN HELP STUDENTS SEE THE BIG PICTURE

In many mathematics classes, chapters are broken into smaller topics that are taught over a span of a few days to a few weeks. Connecting these parts into a coherent whole can be a challenge. Fernandez, Simo, and Sallan (2009), who researched the use of audio podcasts of lessons in an engineering class, found that students repeatedly listened to podcasts to solidify their understanding of the content. Further, they listened most frequently before major assessments to clarify the overarching concepts of the chapters. In a mathematics classroom, students are expected to have more questions before a major assessment; at the same time, teachers want them to develop a coherent view of a chapter's content.

With this research in mind, I created a podcast to help statistics students analyze bivariate numerical data, a multistep process that involves examining numerical and graphical evidence (see **fig. 1**). When approaching this task, students often provide individual parts of the analysis but fail to analyze the data completely. In the eight-minute podcast, each step is detailed verbally, calculator commands are demonstrated, and the final analysis is presented as a sum of all of the individual pieces.

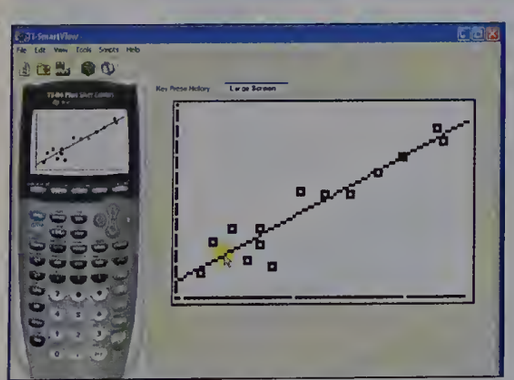


Fig. 1 In this podcast, the graphing calculator is used to assist in a complete analysis of the bivariate data.

The difference between delivering this lesson in class and delivering it as a podcast is that students who view the podcast can start and stop the video at any time to process the information at their own pace, or they can replay certain sections that they did not understand. Thus, the teacher can be more confident that the lesson has been tailored to fit each student's needs.

REASON 2: PODCASTS CAN HELP STUDENTS PACE THEIR LEARNING

Before students can understand overarching concepts, they must navigate through the content presented in daily lessons. Unfortunately, some students are not able to keep the same pace as their teachers and may miss critical components of lessons. Copley (2007) found that students readily downloaded podcasts of daily lessons, especially when the lessons were presented in video format. Students said that the podcasts helped them revise their lecture notes, set a proper pace for learning the content, and catch up on content missed because of absences. Students no longer have to understand every component of a lesson instantaneously but, instead, can fill in the gaps by replaying podcasts of lessons as many times as needed. In particular, a student who may need more time to understand the basic ideas of the lesson can be just as prepared to engage in discussion of the next lesson as can another student who masters the ideas immediately, during the initial lecture.

By the time students enter a statistics course, they should have had multiple opportunities to use the basic rules of probability. Yet some are likely to be exceptionally weak in this area. Thus, a podcast could be assigned the night before the first lesson to ensure that everyone is equally ready to participate. Another possibility is flipping the classroom by disseminating completely new content in a podcast, thus allowing the teacher to focus on discussions, activities, or assignments the following day. These podcasts would be especially useful when introducing particularly difficult topics. Thus, students who may have previously struggled with the fast-paced flurry of new concepts would have an opportunity to engage in classroom discussions designed to help them develop a deeper understanding of the content.

REASON 3: PODCASTS CAN ENRICH LESSONS

In an age when differentiated instruction is encouraged in high school classrooms, enrichment podcasts provide a way to address the needs of high-performing students who may be looking to advance beyond the scope of classroom activities. Walls et al. (2010) examined students' eagerness to use podcasts when learning new content. The researchers created two types of podcasts: those that simply repeated daily lessons and those that enriched daily lessons. Students reported positive attitudes toward using both types and saw the value in using the podcasts to strengthen understanding.

Teachers often participate in events outside school, such as conferences and workshops, that may be valuable to students looking for enrichment beyond the curriculum. Through podcasting, teachers can compose and deliver a portion of this content to these students immediately, thus fulfilling an educational

TIP 2:
You Need Proper Software

Audacity® is a highly recommended free program for recording audio. From the same website, be sure to download the LAME encoder, which will allow you to save your files in the common MP3 format. For video, you can use the free software products CamStudio™ for recording desktop activity and Windows Movie Maker for editing video. Eventually, you may want to upgrade to commercial software that makes the entire process a little more convenient and provides more options for the design of the podcasts.

Plan Your Podcast

Before recording a podcast, write an outline of what you plan to say. Depending on the material and how comfortable you are in front of the camera, you may want to write more of a script. Be careful, however, that you do not end up simply reading to students from a sheet of paper. When your students watch the podcast, they should feel that they are watching you teach a regular lesson in the classroom. Also, creating an outline will help you stay on topic and limit the length of your podcast. Keeping in mind the size of the files and the attention spans of your students, try to keep your podcasts shorter than ten minutes. If they are longer, consider splitting them into multiple podcasts.

need without having to use valuable class time. Also, for students looking for elaborations on in-class topics, such as calculus-based approaches in a statistics classroom or alternative proofs of formulas in a variety of mathematics courses, teachers can use podcasting to provide these advancement opportunities.

REASON 4: PODCASTS CAN PROVIDE REMEDIATION

Because students can replay podcasts, this medium can provide on-demand remediation. In a study by Walls et al. (2010), almost all the students said that podcasts that repeated material taught in class were useful. In a mathematics classroom, lessons often require prerequisite knowledge that students may not have attained. With the podcasts, students can watch or listen to bonus lessons designed specifically to prepare them to handle the in-class content. Further, because this remediation takes place outside the classroom, a student can spend as much time as necessary reviewing the podcast and does not have to struggle in front of peers.

In statistics classrooms, for example, students must frequently use technology for tasks such as generating and graphing data or testing hypotheses. Each of these tasks requires them to follow a chain of keystrokes. Every time teachers introduce a calculator technique that students will need to use in the future, they can create video podcasts using a graphing calculator emulator. Thus, teachers save valuable class time that would otherwise have been spent reteaching keystrokes to students who had forgotten them.

OTHER POTENTIAL BENEFITS OF USING PODCASTS

Because podcasts can be downloaded and viewed on portable electronic devices, this portability could be beneficial to busy students during their on-the-go downtimes (Chan and McLoughlin 2007). Past research, however, has shown that students prefer to view podcasts at home on their computers, and they choose not to multitask during the process (Evans 2008; Lonn and Teasley 2009).

The same is not likely true for current high school students because of the advancement in technology during recent years. As McKinney, Dyck, and Luber (2009) observed, most portable devices have much more memory than previous versions, plenty to hold audio and video podcasts. Further, the devices have larger, clearer screens for viewing video podcasts, thus making it more likely that students will view or listen to the podcasts on their portable devices (see **fig. 2**). Also, portable devices have become more integrated into daily routines and are used much more frequently than only a few years ago.

As part of these new routines, students also frequently receive text messages, social networking updates, and e-mails and feel compelled to check them often. Chan and Lee (2005) suggest that by delivering mathematics content to students through their electronic devices, teachers help students associate the podcasts with other files delivered to students' devices—content that students deem to be important. As a result, mathematics podcasts might be viewed as somewhat socially acceptable by students' peers. Thus, students might be more willing to spend time studying mathematics if they can avoid the social stigma that sometimes comes with reading a mathematics textbook.

In addition to behaviors associated with podcasting, some researchers have focused on its cognitive benefits. O'Bannon et al. (2011) point out that video podcasts are conducive to multichannel processing because they

BRANDON R. HANSON

Fig. 2 Podcasts can be viewed on a variety of portable devices.

simultaneously provide learners with audio and video stimuli, enabling them to make more connections during the learning process. Lazzari (2009) suggests that podcasts can help reduce test anxiety, a mentally crippling ailment for many mathematics students. Finally, Fernandez et al. (2009) suggest that podcasting can enhance the motivation to learn and can help students with different learning styles be equally proficient.

POTENTIAL PITFALLS OF USING PODCASTS

One concern about using mathematical podcasts is that students will opt to use podcasts instead of attending class. Researchers have found little evidence to support this suspicion, and for high school students who have multiple back-to-back classes to skip one particular class seems unlikely (Copley 2007; Lonn and Teasley 2009; Walls et al. 2010; Parson et al. 2009).

Another possible pitfall is that some students may not have access to the equipment needed for watching or

FUZZ MARTIN/ISTOCKPHOTO

Real World Math:
Activities, Lesson Plans, and Articles for the Middle Grades

A New Online Resource for Middle Grades Mathematics Teachers

Teach. Explore. Connect.

Real World Math is an online resource for middle grades mathematics teachers with over 150 articles, lesson plans, and activities, each selected because they connect math to the real world and the real classroom. Access it anywhere—computer, Smartphone, tablet—any device with an Internet connection and browser access.

SPECIAL NCTM MEMBER SAVINGS!
Use code RWMINTRO when placing order for a special introductory price of $19.95.

Explore Real World Math at realworldmath.nctm.org

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

For more information, please call (800) 235-7566 or visit www.nctm.org/catalog

Real World Math
The Biological World Articles
Human Physiology and Health
Zoology
Ecology
Building Percent Dolls: Connecting Linear Measurement to Learning Ratio and Proportion
A series of lessons in which grade 5/6 students use measurement activities to design and construct proportion dolls. This article highlights how measurement can connect learning about percents, decimals, and proportions. Activity ideas are provided for teachers to implement this activity in the mathematics classroom.

listening to podcasts. Teachers must be able to accommodate such students by providing alternative sources for the podcasts or alternative venues for viewing or listening to the podcasts.

A more likely pitfall is a teacher's inability to create podcasts. Two studies have addressed the concern that teachers do not have the necessary skills to create podcasts and distribute them in a way that would benefit their students (Copley 2007; Lonn and Teasley 2009). Teachers must create their own podcasts; an important distinction between podcasts and other online content is that the teacher is creating the podcasts specifically for his or her students. Thus, teachers may need to be trained to create effective podcasts before podcasting can be effective.

TAKE A CHANCE ON PODCASTS

The benefits of podcasting have yet to be proven for undergraduate students and have barely been investigated for high school students. Unfortunately, as Parson et al. (2009) point out, the many confounding variables—including instructor, subject, and format of podcast—make it difficult to measure the effects of podcasting. Combine these variables with the other unique variables associated with high school students in a mathematics course, and it seems likely that the answer will be found only after more subject-specific, age-specific research is conducted. Until then, teachers can either take a chance on the benefits of the amazing technologies in students' pockets or possibly miss an opportunity to make a major impact on students' learning.

Editor's note: For more podcasting information, tutorials, and examples, visit the author's training website at http://www.rhs.rcs.k12.tn.us/teachers/hansonb/podcasttraining/podtraining.htm\.

TIP 4:
You May Need to Convert Your Files

Depending on the methods that you use to create your podcasts, you may need to use a video converter to shrink the size of your files or change them to the proper format. For example, Windows Movie Maker produces files in an AVI format, but iTunes® requires an MP4 format. Thus, you may need to use a free program called Any Video Converter to make this change. This same program can also be used to shrink your file sizes, making it possible to store almost fifty podcasts using less than 2 GB. Another advantage of commercial software is that this feature is often packaged with the program.

REFERENCES

Apple Inc. 2011. "iPod + iTunes Timeline." Apple.com. http://www.apple.com/pr/products/ipodhistory/.

Chan, Anthony, and Mark Lee. September 2005. "An MP3 a Day Keeps the Worries Away: Exploring the Use of Podcasting to Address Preconceptions and Alleviate Pre-Class Anxiety amongst Undergraduate Information Technology Students." Paper presented at the Student Experience Conference: Good Practice in Practice, Charles Sturt University, Wagga Wagga, New South Wales. http://www.csu.edu.au/division/studserv/sec/papers/chan.pdf.

Chan, Anthony, and Catherine McLoughlin. 2007. "Choice, Communication, and Learner Autonomy: The Development of Social Competence Skills Using Podcasting Technologies." In *ICT: Providing Choices for Learners and Learning: Proceedings of Ascilite Singapore 2007*, edited by Roger Atkinson, Clare McBeath, Alan Soong Swee Kit, and Christopher Cheers, pp. 132–34. Singapore: Centre for Educational Development, Nanyang Technological University. http://www.ascilite.org.au/conferences/singapore07/procs/chan poster.pdf.

Copley, Jonathan. 2007. "Audio and Video Podcasts of Lectures for Campus-Based Students: Production and Evaluation of Students' Use." *Innovations in Education and Teaching International* 44 (4): 387–99. doi:10.1080/14703290701602805.

Evans, Chris. 2008. "The Effectiveness of M-Learning in the Form of Podcast Revision Lectures in Higher Education." *Computers and Education* 50 (2): 491–98. doi:10.1016/j.compedu.2007.09.016.

Fernandez, Vincenc, Pep Simo, and Jose M. Sallan. 2009. "Podcasting: A New

TIP 5:
Make Your Podcast Available

After creating your podcast, submit it to iTunes. If accepted, it will be searchable on iTunes, thus making it easier for students to find (see **fig. 3**). Other websites such as YouTube™.com, Facebook.com, and MyBigCampus®.com are possible alternatives because they allow students to be notified instantly when new videos are posted. However, students still have to log on to the site and download the video instead of having it automatically downloaded in iTunes.

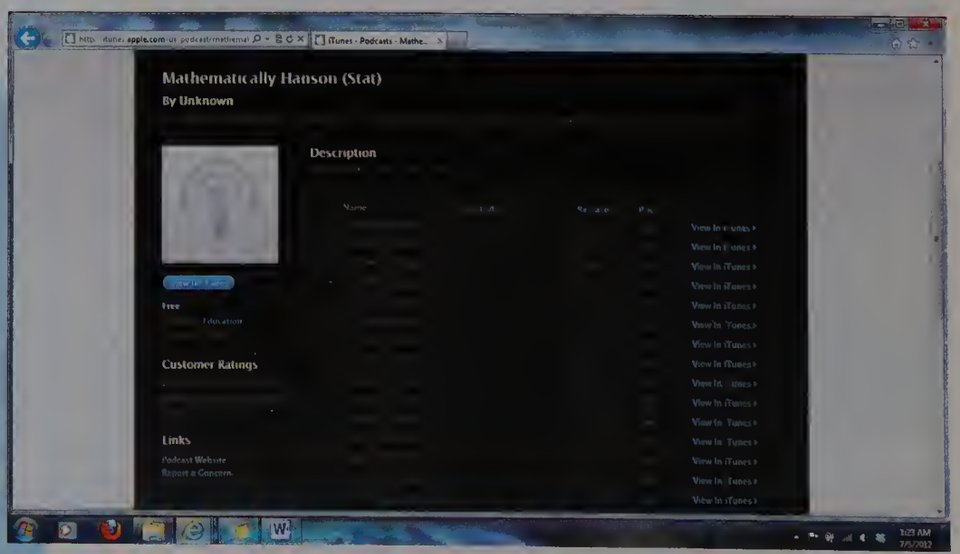


Fig. 3 You can make your podcast searchable through iTunes.

Technological Tool to Facilitate Good Practice in Higher Education." *Computers and Education* 53 (2): 385–92. doi:10.1016/j.compedu.2009.02.014.

Hew, Khe Foon. 2009. "Use of Audio Podcast in K–12 and Higher Education: A Review of Research Topics and Methodologies." *Educational Technology Research and Development* 57 (3): 333–57. doi:10.1007/s11423-008-9108-3.

Lazzari, Marco. 2009. "Creative Use of Podcasting in Higher Education and Its Effect on Competitive Agency." *Computers and Education* 52 (1): 27–34.

Lonn, Steven, and Stephanie D. Teasley. 2009. "Podcasting in Higher Education: What Are the Implications for Teaching and Learning?" *Internet and Higher Education* 12 (2): 88–92. doi:10.1016/j.iheduc.2009.06.002.

McKinney, Dani, Jennifer L. Dyck, and Elise S. Luber. 2009. "iTunes University and the Classroom: Can Podcasts Replace Professors?" *Computers and Education* 52 (3): 617–23.

O'Bannon, Blanche W., Jennifer K. Lubke, Jeffrey L. Beard, and Virginia G. Britt. 2011. "Using Podcasts to Replace Lecture: Effects on Student Achievement." *Computers and Education* 57 (3): 1885–92. doi:10.1016/compedu.2011.04.001.

Parson, Vanessa, Peter Reddy, Jon Wood, and Carl Senior. 2009. "Educating an iPod Generation: Undergraduate Attitudes, Experiences, and Understanding of Vodcast and Podcast Use." *Learning, Media, and Technology* 34 (3): 215–28. doi:10:1080/17439880903141497.

Walls, Stephen M., John V. Kucsera, Joshua D. Walker, Taylor W. Acee, Nate K. McVaugh, and Daniel H. Robinson. 2010. "Podcasting in Education: Are Students as Ready and Eager as We Think They Are?" *Computers and Education* 54 (2): 371–78. doi:10.1016/j.compedu.2009.08.018.

BRANDON R. HANSON, hansonb@rcschools.net, teaches mathematics at Riverdale High School in Murfreesboro, Tennessee, and is pursuing his PhD in mathematics education at Middle Tennessee State University. He is interested in the use of portable digital technologies in education and in developing statistical literacy in teachers and students.

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE NATION'S PREMIER MATH EDUCATION EVENTS

Speak at an NCTM Conference

Want to share your expertise at the **NCTM 2014 Annual Meeting** or one of our **2014 Regional Conferences**? Apply to present next year and join us in New Orleans, Indianapolis, Richmond, or Houston. NCTM conferences help teachers, administrators, and math coaches learn more about challenges facing schools and how to overcome them—especially in effective mathematics instruction. Submit your online proposal to present a session and share your teaching ideas and practices.

Learn more and apply to present at **www.nctm.org/speak**.

2014 ANNUAL MEETING & EXPOSITION
New Orleans • April 9–12
Speakers apply by May 1, 2013

2014 REGIONAL CONFERENCES & EXPOSITIONS
Indianapolis • October 29–31
Richmond • November 12–14
Houston • November 19–21
Speakers apply by September 30, 2013

Christopher V. Cappiello

A Closer Look at Function Transformations

High school mathematics classes today often approach graphing and functions simply by presenting the graphs of certain basic functions and describing transformations that can be performed. Start with the graph of $y = x^2$, a parabola. The basic transformations of this graph involve stretches and translations. For example, changing the equation to $y = x^2 + 3$ results in a vertical translation of 3 units, and a change to $y = (x + 3)^2$ results in a horizontal translation of 3 units to the left. The equation $y = 3x^2$ results in a vertical stretch, and $y = (3x)^2$ produces a horizontal shrink (or stretch by a factor of 1/3).

Such transformations can be applied to virtually any function, not just $y = x^2$. For a simple function like $y = x^2$, transformations can be combined fairly easily, and any variation on the simple quadratic function can be graphed just as easily by interpreting these transformations. However, trying to approach more complicated functions this way is often difficult, and we often have to resort to using a graphing tool.

I present a series of procedures that allows anyone to create a fairly accurate approximation of the graph of a complicated function without using a calculator. This method involves looking at complicated functions as simple functions combined using addition, multiplication, and composition. For example, the graph of $y = |x + 4|$ is the graph of $y = x + 4$ with the portion below the x-axis reflected above it. This explanation shows why the 4 inside the absolute value symbol appears to cause a horizontal rather than a vertical shift (the location of the vertex of the graph of $y = |x + 4|$ is located at the x-intercept of the graph of $y = x + 4$).

The graph of $y = x^2 + 3x$ can be seen as the sum of the graphs of $y = x^2$ and $y = 3x$; breaking the function into its component parts helps explain the

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

Edited by ***Maurice Burke****, Maurice.Burke@utsa.edu*
University of Texas at San Antonio

J. Kevin Colligan*, jkcolligan@verizon.net*
RABA Center of SRA International, Columbia, MD

Maria Fung*, mfung@worcester.edu*
Worcester State College, Worcester, MA

Jeffrey J. Wanko*, wankojj@muohio.edu*
Miami University, Oxford, OH

location of the vertex and x-intercepts. This method helps wean students from an overreliance on graphing calculators and prevents them from making generalizations that can lead to mistakes.

THE INSPIRATION

The inspiration for this method was a problem that my fellow student J. T. Kucera posed and that my mathematics teacher, Ron Garrett, passed on to me:

> Consider the difference between the graphs of $y = |2x - |3x + 6||$ and $y = |2x - |3x + 6| + 4|$. (See **fig. 1**.)

How should the +4 term in the second function affect the graph? The first two suggestions might be a horizontal shift 4 units left or a vertical shift 4 units up, but neither is true. Instead, the effect of the +4 term is a vertical translation but *down* 4 units (see **fig. 1b**).

Explaining the effect of the +4 term using the rules that students are most familiar with is difficult. However, the effect becomes apparent when we break the composite function into its basic component functions.

First, consider the simpler graph of $y = |x + 4|$. The typical interpretation is that the +4 term in the function represents a horizontal translation of the basic absolute value function to the left—essentially, a true statement. The more complicated example above, however, cannot be solved without understanding what causes this horizontal translation. The graph of $y = |x + 4|$ is the graph of the absolute value of $x + 4$, so the part of the graph of $y = x + 4$ that lies below the x-axis is reflected above the x-axis. **Figure 2** shows both the original line and the reflection of the part below the x-axis across that axis.

So the apparent translation to the left compared with the basic absolute value graph results from the graph of $y = x$ being shifted up 4 units, a move that changes the x-intercept before the absolute value is considered. If the slope of the line were greater, smaller, or even negative, the x-intercept would change, and thus the location of the vertex in the absolute value graph would change. To locate the vertex of the graph of an equation such as $y = |2x - 6|$, we must divide the constant term by the coefficient of x (the x-coefficient is the slope).

Returning to our original problem, we can ask why, in the function $y = |2x - |3x + 6| + 4|$, the 4 at the end appears to translate the graph down. Consider the innermost function in the composite function and work outward. The absolute value of $y = 3x + 6$ produces a relatively tall and narrow absolute value graph, apparently translated to the left by 2 units. The negative sign in front of $|3x + 6|$ makes

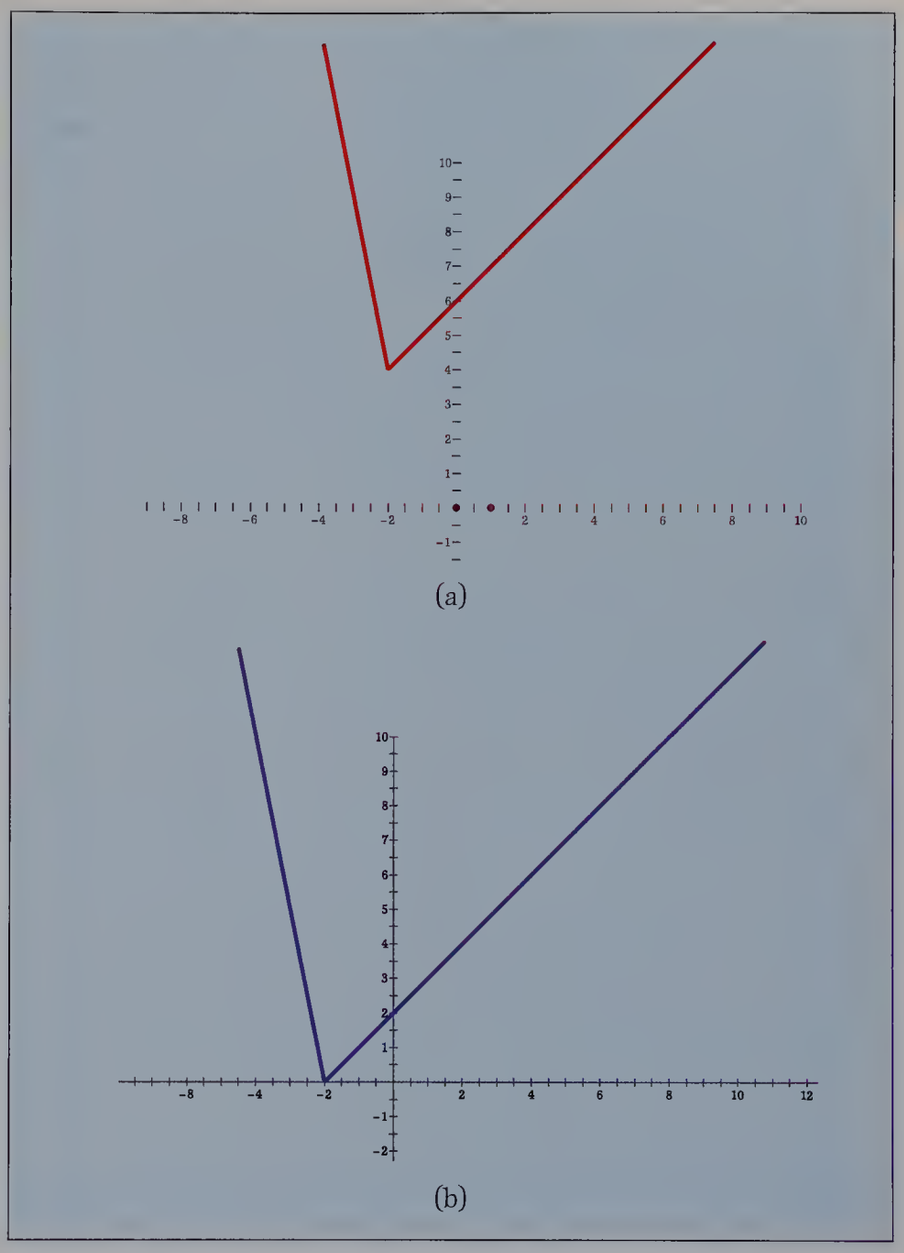


Fig. 1 Comparing the graphs of $y = |2x - |3x + 6||$ (**a**) and $y = |2x - |3x + 6| + 4|$ (**b**) yields a surprise.

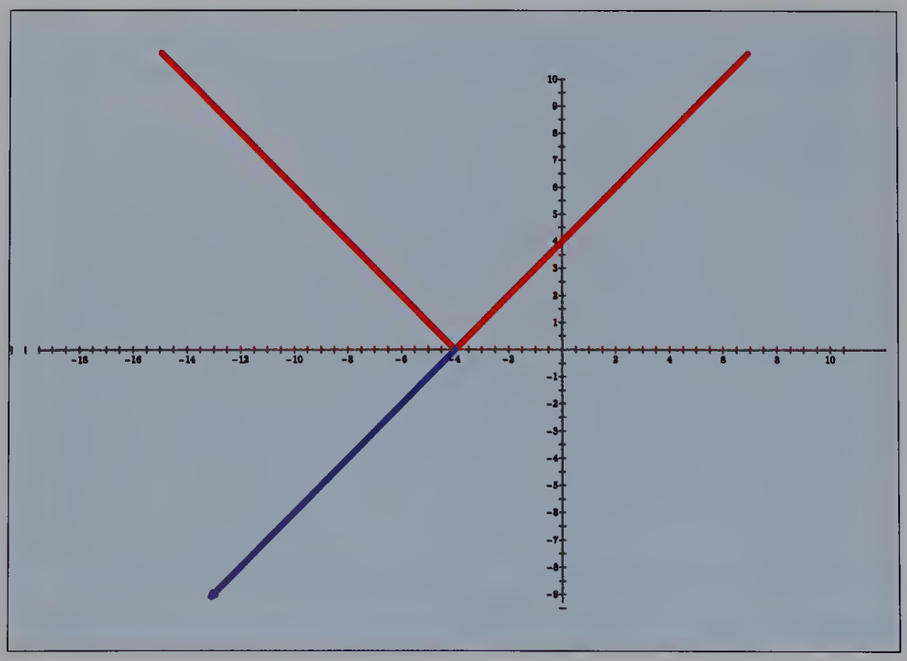


Fig. 2 The graphs of $y = x + 4$ and $y = |x + 4|$ are plotted on the same axes.

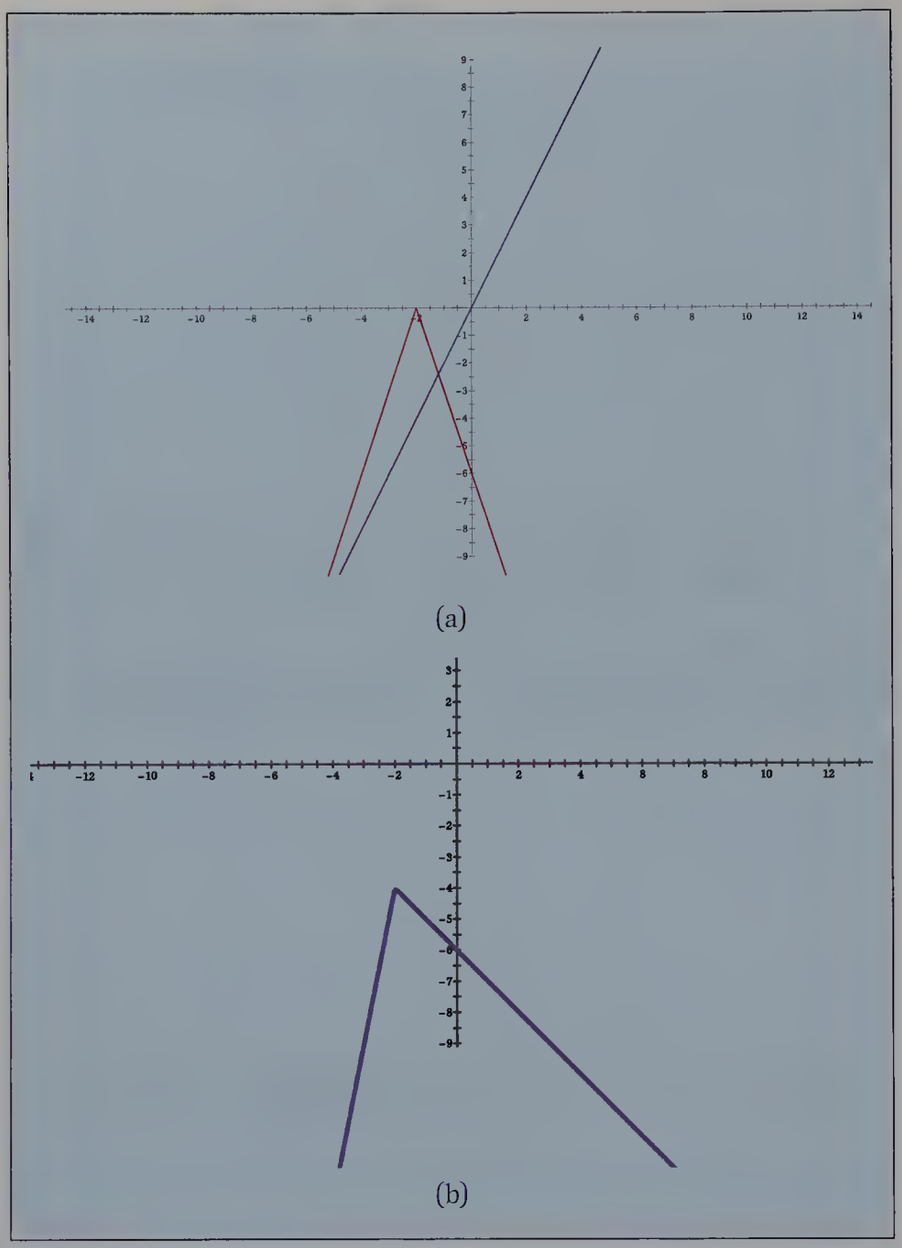


Fig. 3 The functions $y = -|3x + 6|$ and $y = 2x$ are graphed separately (**a**) and then as a sum (**b**).

every y-value negative, reflecting the graph across the x-axis. The function then says to add $2x$. These two individual graphs are shown on the same set of axes (see **fig. 3a**), and the resulting graph is shown in **figure 3b**.

The addition of a linear function to an absolute value function is responsible for the asymmetrical look of the graph. (On the left branch of the graph in **fig. 3b**, the slope is steep, but on the right branch it is –1, the result of adding two positive slopes on the left while adding both a positive and a negative slope on the right.) More important, the reason that the +4 term translates the final graph down becomes apparent. If the +4 term is omitted, all that remains to complete the graph is to reflect the graph in **figure 3b** across the x-axis because of the absolute value. But inserting the +4 term within the absolute value moves the whole graph closer to the x-axis before the flip, resulting in the final graph being lower.

Although this problem may not seem particularly useful, it is just one example of a graph that can be understood by breaking it down and graphing each of its component parts.

ANALYZING THE METHOD

Similar reasoning can be applied to the graph of virtually any type of function. The two simplest functions are constant functions ($y = b$) and linear functions ($y = mx + b$, $m \neq 0$). Linear functions are combinations of a linear term and a constant. But viewing each term as a function itself is useful. In any function that includes multiple nonlike terms, each term is itself a function of the variable, and the value of their sum (i.e., y) is found by adding the ordinates of these simpler component functions. In other words, the function $f(x) = mx + b$ could be written as $f(x) = g(x) + h(x)$ where $g(x) = mx$ and $h(x) = b$. This decomposing may be oversimplification when working with linear functions, but it is very useful when dealing with the graphs of other types of functions.

Consider, for example, the graph of $y = x^2 + 3x$. We know that the vertex of this parabola has x-coordinate $-b/(2a) = -3/2$, and we can use that to calculate the y-coordinate to obtain

$$V\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

Consider this explanation: First, break this function into $y = x^2$ and $y = 3x$ and graph both components (see **fig. 4**). The graph shows that when the linear function is added to the quadratic function, every point on the right side of the parabola will move up and every point on the left side will move down (because the value of the linear function is positive when x is positive and negative when x is negative).

The problem then becomes how to represent the resulting graph accurately. In general, when graphing a quadratic function such as this, the vertex, both x-intercepts, and perhaps the y-intercept could be used to give an accurate sketch. We note that when two functions are added, the slope of the resulting function at any x-value is equal to the sum of the slopes of the two original functions at that same x-value. (A simple illustration: $y = 2x + 3x$ is equivalent to $y = 5x$.)

Although a parabola does not have a constant "slope," we can use the derivative to calculate the slope at any desired point. Thus, the vertex (where the derivative is zero) of $y = x^2 + 3x$ can be found where the slope of $h(x) = 3x$ and the derivative of $g(x) = x^2$ are opposites (because x^2 and $3x$ are the terms of the function of interest). So, for the general quadratic $ax^2 + bx + c$, let $g(x) = ax^2$ and

let $h(x) = bx + c$. The vertex occurs where $g'(x) = -h'(x)$—that is, for x-values such that $2ax = -b$ and, thus, when $x = -b/2a$. This x-value can then be substituted into the original function to find the associated y-value.

The x-intercepts (0 and −3) can then be determined by finding where $g(x) + h(x) = 0$. We generate a graph of the combined function (see **fig. 5**) by interpreting what the function's formula is saying.

I have laid out some of the basic rules for what traditionally has been called *addition of ordinates.* (For more on addition of ordinates, see, e.g., Riddle [1979, 1996].) But what about the product of functions or multiplication of ordinates?

WHEN COMPONENTS ARE FACTORS

When multiplying functions, the first thing to do is find the zeros and discontinuities of the two functions; these are potential zeros or discontinuities of the product function. Next, find the points where each function equals 1; at these x-coordinates, the value of the product function will be the value of the other function. We can also find the points where each function equals −1—that is, where the value of the product function will be the opposite of the value of the other function. Once we have these values, creating a rough sketch should be relatively easy.

For example, consider the function $y = x \sin x$. The function $f(x) = x$ has one zero at $x = 0$, equals 1 at $x = 1$, and equals −1 at $x = -1$. The function $g(x) = \sin x$ has zeros at $x = 0 + k\pi$, equals 1 at $x = \pi/2 + 2k\pi$, and equals −1 at $x = 3\pi/2 + 2k\pi$, k an integer. Because $g(x) = \sin x$ has more points where $g(x) = 1$ or −1, plotting these values rather than the 1 and −1 values for $f(x) = x$ will be more useful when making a rough sketch of the product function.

Figure 6 shows $g(x) = \sin x$ and the product function. Note that the functions have zeros at the same set of x-values. A visually guided multiplication of ordinates will not produce a perfect graph, but it will yield a rough sketch. The graph in **figure 6** shows a progressive vertical stretch; that is, the extrema approach $+\infty$ or $-\infty$ as x approaches $+\infty$ and $-\infty$. This observation becomes useful in sketching the graphs of certain functions.

In the example $y = x \sin x$, if we multiply the y-value of any maximum or minimum of $y = \sin x$ by the x-value at that point, the product will be the y-value of a point on the graph of $y = x \sin x$ that is tangent to either $y = x$ or $y = -x$. In fact, the graph of $y = x \sin x$ is bounded above by the graph of $y = |x|$ and bounded below by the graph of $y = -|x|$. So the lines $y = x$ and $y = -x$ can be used as guides to graph $y = x \sin x$ and to approximate its extrema.

Similarly, the function $y = x^2 \sin x$ is bounded by the parabolas $y = x^2$ and $y = -x^2$. In general, when

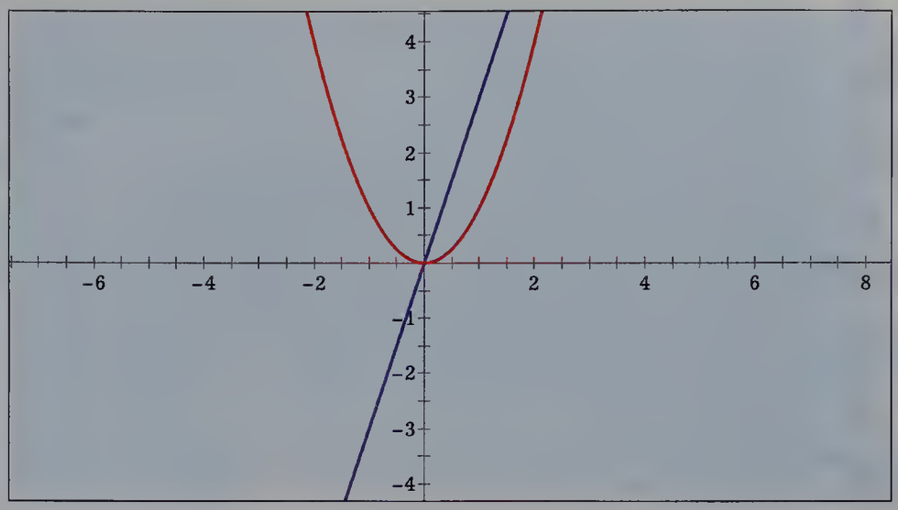


Fig. 4 The functions $y = x^2$ and $y = 3x$ are plotted on the same axes.

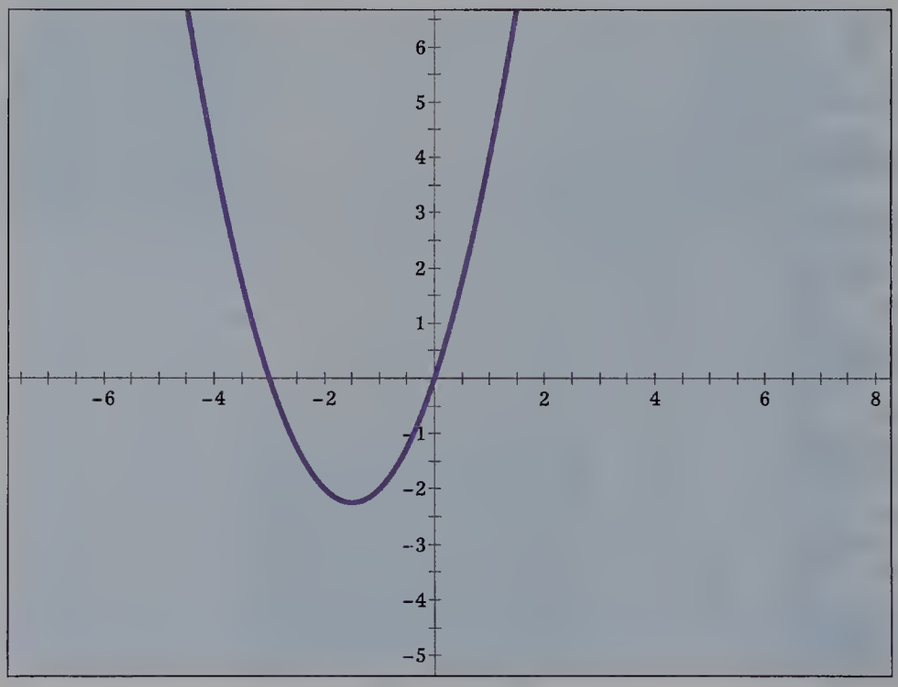


Fig. 5 The graph of $y = x^2 + 3x$ can be obtained by summing two simpler functions.

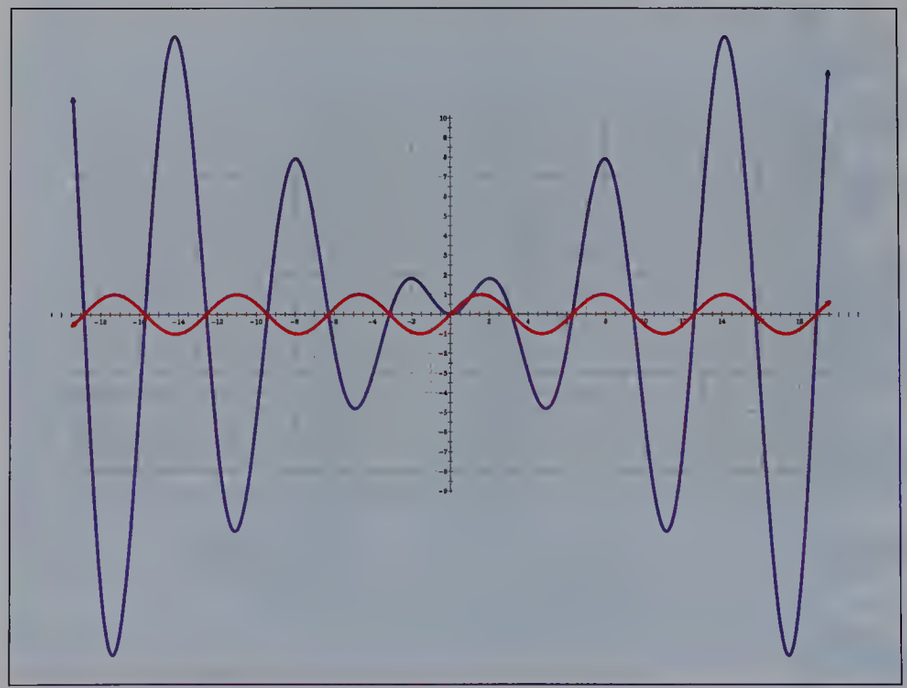


Fig. 6 The functions $y = \sin x$ and $y = x \sin x$ are plotted on the same axes.

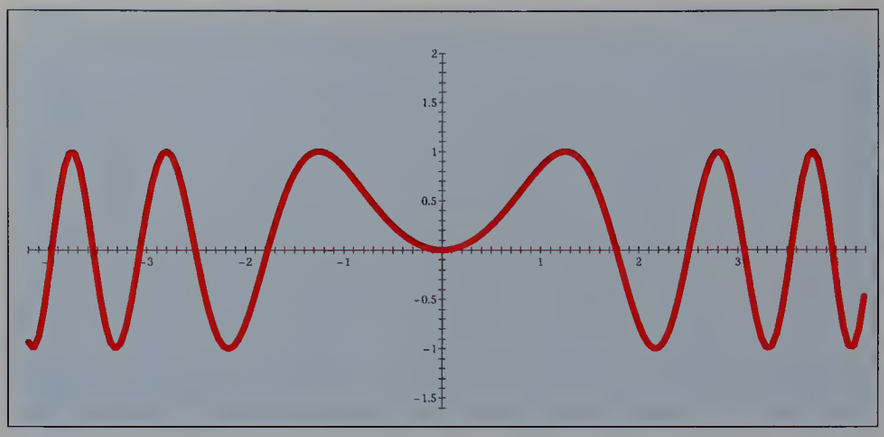


Fig. 7 The graph of $y = \sin(x^2)$ shows a progressive horizontal shrink as x approaches infinity and negative infinity.

a bounded sinusoidal function (e.g., the sine and cosine functions) is multiplied by a monomial function $f(x) = x^n$, the sinusoid is bounded above by the graph of $y = |x^n|$ and bounded below by the graph of $y = -|x^n|$.

Similar principles can be applied to composite functions of the form $f(g(x))$. The zeros of such functions are found wherever the "interior function" $g(x)$ produces a zero of the "exterior function" $f(x)$—in other words, wherever $g(x)$ can be substituted into $f(x)$ to yield a value of zero. For example, to find the zeros of $y = \sin(x^2)$, we must simply find the points for which $x^2 = k\pi$.

An analysis of the graph of $y = \sin(x^2)$ produces an interesting result, which will be useful for graphing similar composite functions. Again, we begin by locating the zeros, which occur wherever $x^2 = k\pi$, k an integer, so we will turn our attention momentarily to the function $y = x^2$. For $x > 0$, $y = x^2$ is increasing, and its graph is concave up, meaning that if we list points on the graph that have arithmetically increasing y-values, the difference between their x-values decreases with each successive point. In algebraic terms, if $y = x^2$ for $x > 0$ is rewritten as $x = y^{1/2}$, as y increases linearly, x increases at a declining rate.

Thus, turning back to the graph of $y = \sin(x^2)$, we see that points where $x^2 = k\pi$ occur closer together as x approaches $+\infty$ or $-\infty$. The result in the graph of $y = \sin(x^2)$, and in any composite function where the magnitude of the slope of the interior function increases as x approaches $+\infty$ or $-\infty$, is a progressive horizontal shrink that becomes more pronounced as x approaches $+\infty$ or $-\infty$ (see **fig. 7**).

To illustrate the reverse effect, consider the function $y = \sin(1/x)$. As x approaches $+\infty$ or $-\infty$, the graph of $g(x) = 1/x$ approaches the line $y = 0$. So for large $|x|$, the graph of $y = \sin(1/x)$ approaches $\sin 0 = 0$.

As an extension of the methods discussed here, students may be challenged to investigate horizontal and vertical asymptotes of complicated functions created by adding, multiplying, and composing simpler functions.

A DEEPER UNDERSTANDING

If students learn *why* something works, not just that it *does* work, their understanding has more depth. In my experience, one of the biggest reasons many students dislike mathematics is its seeming arbitrariness. They learn that a certain function has a particular graph related to it or that an alteration to that function changes the graph in a certain way, but they do not always understand why.

This analysis of function graphing helps students understand the behavior of functions by examining the underlying mathematical components. In my original example—the difference between the graphs of $y = |2x - |3x + 6||$ and $y = |2x - |3x + 6| + 4|$—it becomes clear that the +4 term cannot be assumed to produce a horizontal shift, even though in a simpler function (e.g., $y = x^2$ vs. $y = [x + 4]^2$) it would. This sort of reasoning should be taught alongside regular graphing techniques both as a justification for the look of a given graph and as a method of graphing without having to use a calculator.

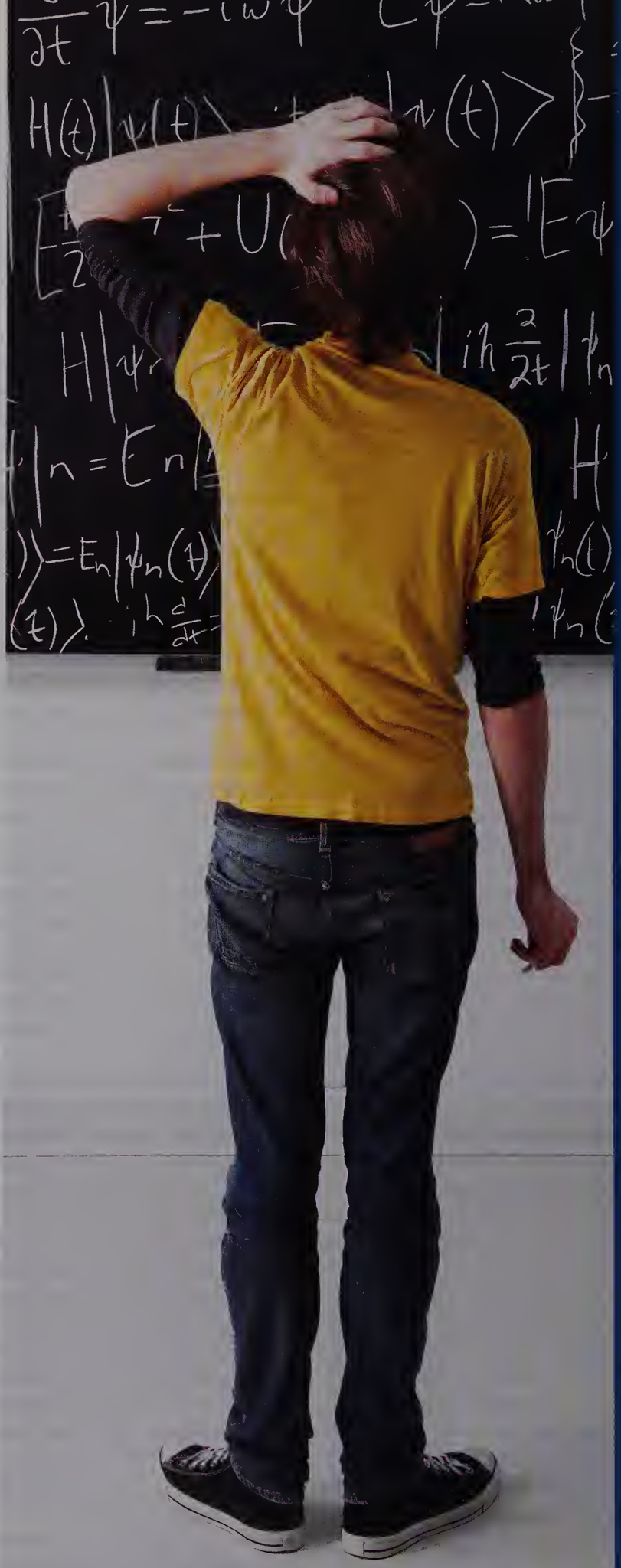
REFERENCES

Riddle, Douglas F. 1979. *Calculus and Analytic Geometry*. 3rd ed. Belmont, CA: Wadsworth.

———. 1996. *Analytic Geometry*. 6th ed. Boston: PWS.

CHRISTOPHER V. CAPPIELLO, christopher.cappiello@yale.edu, is a physics major at Yale University. He is interested in applying geometry and graphical techniques to problem solving. He began writing this article when he was a junior in high school and submitted it during his senior year.

Reasoning and Sense Making



In *Focus in High School Mathematics: Reasoning and Sense Making* (2009), NCTM advocates that all high school mathematics programs have at their core reasoning and sense making. *Reasoning* is "the process of drawing conclusions based on evidence or stated assumptions," and *sense making* is "developing understanding of a situation, concept, or context by connecting it with existing knowledge."

The *Mathematics Teacher* Editorial Panel solicits manuscripts that address the following questions related to reasoning and sense making:

// How does a teacher generate a culture of reasoning and sense making in the classroom?
// What issues of equity arise, and how can classroom teachers fairly address them?
// What sorts of tasks help students progress to higher levels of reasoning?
// How can we teach students to evaluate the reasonableness of their processes and results?
// How do we assess or evaluate student reasoning and sense making?
// How can we help stakeholders—teachers, parents, curriculum authors—understand the significance of reasoning and sense making?
// How do we address reasoning and sense-making challenges that students encounter?
// How does a teacher integrate reasoning and sense making with the mathematical practice and content of the Common Core State Standards?

Submit your completed manuscript for review to the *Mathematics Teacher* at **mt.msubmit.net**. Be sure to check the box indicating that this manuscript is for the Reasoning and Sense Making call. Guidelines for the preparation of manuscripts can be found at nctm.org/publications/content.aspx?id=22602.

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

CALL FOR MANUSCRIPTS

AUTHOR GUIDELINES for *Mathematics Teacher*

Writing about your work and getting published can be a rewarding personal and professional development experience. Following are author guidelines formulated by the *MT* Editorial Panel.

QUALITATIVE CONSIDERATIONS

- Preference will be given to articles with the potential for direct classroom application, especially those providing evidence of classroom practice (including discussion of classroom activities, photographs, or examples of student work).
- Manuscripts from classroom teachers are particularly encouraged. If you wish to discuss your article ideas before submitting a manuscript, contact the journal editor at agoetz@nctm.org.
- Consider submitting a manuscript for one of the current calls for manuscripts; these can be accessed at www.nctm.org/publications/content.aspx?id=8756.
- Manuscripts containing original derivations or proofs without specific links to the high school classroom will be referred to the Reader Reflections department (if their length is appropriate).
- Aim for an informal style that appeals to *MT* readers (review recent issues of the journal for examples).
- Manuscripts that feature the use of technology to teach and learn mathematics should be as generic as is reasonable in describing products.
- Proofread your manuscript in advance for English usage, mathematical correctness, and accuracy of references. Use a spell checker before submitting your manuscript and recheck it for grammar, completeness, spelling of all proper names, and full identification of all organizations and groups named by initials or acronyms.

TECHNICAL CONSIDERATIONS

Preparation of Manuscripts

1. Double-space all material, including quoted matter, lists, tables, notes, references, and bibliographies. Manuscripts should not exceed 2500 words (ten pages), not including tables and figures. Include only tables and figures essential to understanding.
2. Put the manuscript title and name(s) of author(s) on a separate cover sheet. No author identification should appear in the manuscript. Do not reference your own work in a way that compromises the blind-review process.
3. Provide accurate and complete bibliographical information. All references cited in the manuscript should be listed at the end of the manuscript. See the *Chicago Manual of Style,* 16th ed. (Chicago: University of Chicago Press, 2010), for additional details.
4. Number all figures and tables; these may appear where appropriate in the text. Refer to them by number—for example, "see figure 1." *Include captions for all figures and tables; captions must be complete sentences.* The journal designer will prepare and position figures and artwork, so statements such as "insert figure 3 about here" are unnecessary.
5. *Do not use special formatting* (color, underlining, text wrapped around art, etc.). All such formatting must be deleted before the manuscript is typeset.
6. Do not use footnotes. Integrate whatever text would be in a footnote into the body of the manuscript.
7. Avoid using the MS Word equation editor. Do not use MathType for simple variable or function names or expressions. Expressions such as $f(x)$ or $a > 3$ should be typed using standard word-processing keys, while $f(x) = e^{\wedge}(\text{sqrt}(x^{\wedge}2 + 1))$ should be created using MathType to obtain $f(x) = e^{\sqrt{x^2+1}}$.
8. Use geometric symbolism and terminology appropriately. In particular, distinguish among segments, lines, rays, and angles and the measure of a segment or angle.
9. Very short programs for programmable calculators may be published to support an activity contained in an article.

Quotations, Photographs, Student Work, and Permissions

10. Copy quoted material exactly. Do not change any words or punctuation. Mark each omission by ellipsis points and put each interpolation in brackets. Indicate the author's italics by underscoring the words. If you want to add italics yourself, insert the phrase "(italics added)" or "(italics mine)."
11. For every quotation, give the source in full; provide a specific page number for each direct quote. Likewise, credit all photographs and drawings and reference conference presentations.
12. You must obtain permission to quote copyrighted material and reproduce copyrighted pictures or illustrations. Ordinarily, it is not necessary to obtain permission to quote a passage of less than 100 words from a long prose article or book, but permission is needed for more than a single line of poetry and for more than 25 words of a short prose composition. It is the author's responsibility to obtain the appropriate written permission for the use of any copyrighted material in his or her manuscript.
13. Submit photographs and student work electronically along with the manuscript. Obtain parental permission if a photograph clearly shows the student's face (NCTM journals style is to delete students' full names in text and on worksheets).

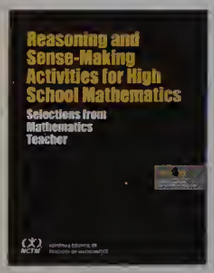
All manuscripts must be submitted electronically to *Mathematics Teacher* at this site: **http://mt.msubmit.net**. An electronic acknowledgment will be sent on receipt of a manuscript.

FOR YOUR information

PUBLICATIONS

From NCTM
Individual NCTM members receive a 20 percent discount on NCTM publications. To order, visit the NCTM online catalog at www.nctm.org/catalog or call toll free at (800) 235-7566. Free print catalogs of NCTM publications also are available by writing to NCTM.

Reasoning and Sense-Making Activities for High School Mathematics: Selections from *Mathematics Teacher*, Sarah Kasten and Jill Newton, eds., 2012. Grades 9–12. 344 pp., $39.95 paper. ISBN 978-0-87353-655-4. Stock no. 13784. National Council of Teachers of Mathematics; www.nctm.org.



I am always on the lookout for new problems and activities to engage my students. Back in 1999, for example, *MT* published an article by Don Gernes ("The Rules of the Game," *MT* May 1999, vol. 92, no. 9, pp. 424–29) that included an activity based on a simple deductive system discussed in Douglas Hofstadter's book *Gödel, Escher, Bach*. As it turns out, this great activity has stood the test of time—the article is included in this collection of articles and activities.

A short but valuable introduction outlines the influence of NCTM's earlier publication *Focus in High School Mathematics: Reasoning and Sense Making* (2009). Each of the five chapters—on number and measurement, algebraic symbols, functions, geometry, statistics and probability—includes eight or nine articles; each begins with an introduction and an activity matrix with column headings for author and title, mathematical topics, contexts, and materials. Each article is accompanied by a set of activities. A companion website contains PDFs of each activity page so that making class sets is easy. Appendix A lists each article with the corresponding topic clusters from the Common Core State Standards of Mathematics. The companion website also contains another two dozen *MT* articles that could be used to develop activities; these are listed in Appendix B. The activity sheets are the core of this valuable resource. Many of the selected *MT* articles were published with activity sheets; however, for other articles, activity sheets had to be created for this collection.

A committee of about a dozen secondary school mathematics educators nominated articles for this book. Most were originally published in the last twenty-five years, but some from 1946, 1955, and a few from the early 1970s indicate a thorough search job. The breadth of topics seems very good. High school teachers should be able to find activities related to almost any standard high school mathematics class, with the notable exception of calculus. The authors appear to have made a decision to follow the Common Core Standard's lead and not include calculus activities.

The book can be used by new teachers, by teachers who are just beginning to venture away from using a single mathematics textbook, and by experienced teachers who have developed their own classroom reasoning and sense-making activities. The combination of an article and a PDF for each activity will be appreciated by all teachers.

I wish the book said a little about how one goes about crafting a good activity. Experienced mathematics teachers understand the difference between an exercise and a problem; they know that a given question might be an exercise for one group of students but a problem for another. Over time, a good teacher has students working on an appropriate selection of exercises and problems. Each activity has a good mixture of challenge, scaffolding and open-ended questions for some high school classes, although this mixture might not be the best for a particular class. In the introduction, Kasten and Newton offer nine good tips for facilitating an activity. At least two of these should apply to the design of activities as well: (1) "Provide tasks that require students to figure things out for themselves" and (2) "Resist the urge to tell students how to solve a problem when they become frustrated; find other ways to support students as they think and work."

This book is a valuable resource for high school mathematics teachers. Every high school mathematics department should have a copy.

—Bruce Cohen
Lowell High School
San Francisco

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

FROM OTHER PUBLISHERS

The Best Writing on Mathematics 2011, Mircea Pitici, ed., 2011. Foreword by Freeman Dyson. 414 pp., $19.95 paper. ISBN 978-0-691-15315-5. Princeton University Press; www.press.princeton.edu.

This collection of twenty-six well-researched, scholarly articles focuses on the nature of mathematics and the applications of mathematics to real life and other situations. The articles deal with the applications of mathematics to a variety of disciplines, including art, education, economics, sculpture, medicine, physics, warfare, and philosophy. From

these writings, it is evident that mathematics is everywhere. With a few exceptions, the articles are relatively short, thus making reading easy.

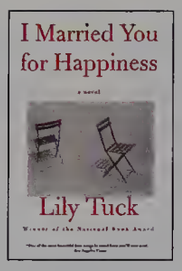
The prevalence of mathematics in uncommon situations is revealing. For example, some readers will be amazed, as I am, at the amount of mathematics in the Rubik's cube, the art of M. C. Escher, the magnetic resonance image (MRI), sculpture, games (e.g., the Duckworth-Lewis system in cricket), and such problems as matching medical students to hospitals. Mathematics educators, including teachers, should find the discussions on mathematics education, mathematical tables, autism, the development of computers, problem solving, mathematics as a way of thinking, proof, number sense and other aspects of numbers particularly interesting because of their direct relevance to the teaching of mathematics.

An elaboration of some of the issues would have been helpful to the average reader; Fermat's last theorem could have been stated, and an index, for easy referencing, would have been appropriate. Because the topics are so varied and some are highly specialized, reading the book from cover to cover will not be meaningful to readers without extensive knowledge of many topics. Readers without an adequate understanding of philosophical theories, for example, are unlikely to make much sense of the writings on the philosophy of mathematics or of the sections on physics. Readers must choose articles to read according to their interest and educational background.

Mathematicians, mathematics educators, and social and natural scientists will find this book illuminating.

—Emam Hoosain
Augusta State University
Augusta, GA

I Married You for Happiness, Lily Tuck, 2011. 193 pp., $24.00 cloth. ISBN 978-0-8021-1991-9. Atlantic Monthly Press; www.groveatlantic.com.



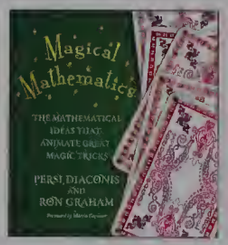
"Was he happy? Did she make him happy?" On the night of his death, Nina eulogizes her husband, sifting through memories of forty-three years of life and love. Nina, an artist, and Philip, a mathematics professor, begin their journey together in France and return every summer. With this constancy, Nina remembers life's twists and turns through the voice of her husband's probability theory lectures from both private conversation and the public classroom. The likelihood of an affair, his daughter's hair color, and the existence of an afterlife are posed as applications of mathematics. These lectures reveal Philip's appeal as a husband and beloved teacher as well as his irritating confidence.

The novel explores life's truths—life and love, joys and disappointments—through a deeply imperfect marriage with many secrets. Recalling her life in the shadows of unappreciation, Nina examines the question of her own happiness in the marriage, coming to no clear conclusion.

The mathematics of the novel is simple and approachable for readers with little or no passion for the subject. This book is intended for pleasure rather than classroom use.

—Estelle M. Lockhart
Math Solution Tutoring
Miami, FL

Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks, Persi Diaconis and Ron Graham, 2011. Foreword by Martin Gardner. 258 pp., $29.95 cloth. ISBN 978-0-691-15164-9. Princeton University Press; www.press.princeton.edu.



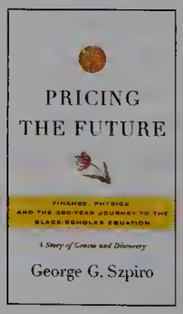
Diaconis and Graham present engaging magic tricks based on interesting mathematics. The tricks—involving playing cards, ropes, paper folding, juggling, and more—include detailed instructions and an explanation of the underlying mathematical concepts. One attractive feature is the emphasis on the history of mathematics and the current work of practicing mathematicians.

A gifted and talented ten-year-old, intrigued by the card trick using Hummer shuffles, borrowed my copy of the book, followed the step-by-step instructions (clarified by accompanying illustrations), and impressed his school friends with the magic trick. Certainly, the formal proofs of the theorems and corollaries were beyond his interest and comprehension, but perhaps the trick will foster a future interest in mathematics.

The simple tricks, like juggling balls, have connections to mathematical concepts, but the proofs are challenging for the average reader. The tricks are engaging avenues to discussion of mathematical applications for those teaching or studying advanced mathematics. The magic tricks can be learned quickly but will require time to perfect. Even readers with a strong mathematical background may have difficulty understanding the related mathematics.

—Christie Bledsoe
University of Mary Hardin-Baylor
Belton, TX

Pricing the Future: Finance, Physics, and the 300-Year Journey to the Black-Scholes Equation: A Story of Genius and Discovery, George G. Szpiro, 2011. 320 pp., $28.00 cloth. ISBN 978-0-465-02248-9. Basic Books; www.basicbooks.com.



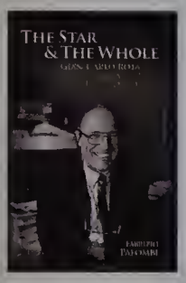
Pricing the Future is an enjoyable and intellectually stimulating read. Szpiro creates a very readable story that examines the historical development of the formula giving the true value of a financial option. The development of this formula, by Fischer Black, Myron Scholes, and Robert Merton, is considered of critical importance to people wishing to understand financial markets. For their work, Scholes and Merton received the 1997 Nobel Prize in Economics (Black died in 1995).

The story focuses not only on the people involved in the historical development of the formula but also on the connection of scientific theories such as the movement of atoms and molecules to the movement of stock prices. The book looks at how people from fields such as physics and mathematics all played a role in the development of the formula.

Although this book is more suited for serious finance professionals and students, it has importance for mathematics educators. It clearly shows how ideas from different domains connect to and influence one another—a very important theme for mathematics teachers.

—James Pelech
Benedictine University
Lisle, Illinois

The Star and the Whole: Gian-Carlo Rota on Mathematics and Phenomenology, Fabrizio Palombi, 2011. 138 pp., $29.95 paper. ISBN 978-1-56881-583-1. CRC Press/Taylor & Francis Group; www.crcpress.com.



This short book on Gian-Carlo Rota and his work in phenomenology in mathematics was written by a student of Rota's. The book is divided into four sections: biographical information and the historical context of Rota's work; an overview of Rota's phenomenology, based on previous work by Husserl; a discussion of how phenomenology fits with mathematics; and phenomenology's connections with the value and importance of mathematics within the larger scientific community.

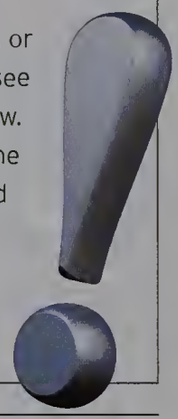
Palombi, writing from a first-person perspective, assumes that readers are familiar with Rota's work in phenomenology and mathematical philosophy. The book is heavily footnoted with numerous primary sources, many of which are Rota's, Heidegger's, or Husserl's. The academic style is one that only a narrow audience will understand—those who spend considerable time working in mathematical philosophy and those knowledgeable about the writings of these mathematicians. For high school mathematics teachers wishing to broaden their recent mathematical history, this book will challenge their thinking while expanding their mathematical vocabulary.

This book would be a valuable library addition for mathematical philosophers and many mathematics teachers. Unfortunately, students below the graduate level will be challenged to understand the book's full intent. The language is highly technical and far above what high school students and even some college students can handle.

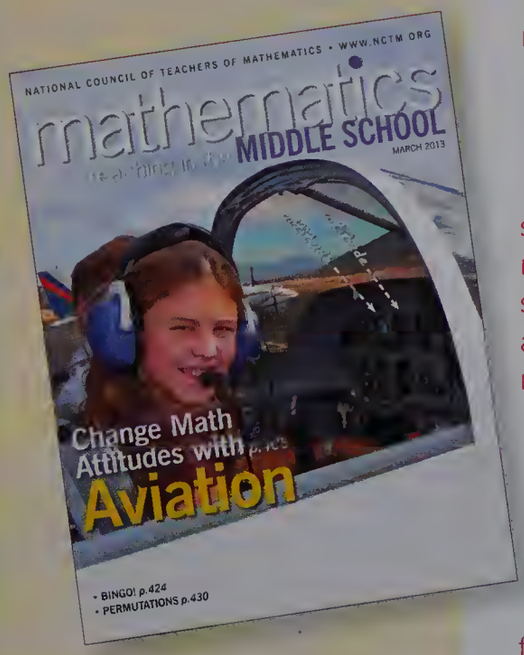
—Jonathan Engelman
Kettering College
Kettering, OH

GIVE US YOUR SUGGESTIONS

If you know of a new book or product that you wish to see reviewed in *MT*, let us know. Send information about the item, including its title and the publisher's name, to mtreviews@nctm.org. We will be sure to consider the item for FYI.



The *MTMS* Word Problem



The Editorial Panel of *Mathematics Teaching in the Middle School* is asking readers to help solve the *MTMS* Word Problem:

Which word or words should be used in a new journal title?

Just as *Mathematics Teacher* and *Teaching Children Mathematics* are descriptive without using the labels "high school" or "elementary school," *MTMS* might better serve a wider range of educators without the words "middle school." What better source for a new title than our readers? Get creative, and send us your favorite names. The rationale of the Editorial Panel and more information are available online at **www.nctm.org/mtmswordproblem**.

Submit your most creative names through **www.nctm.org/mtmswordproblem** by **May 15, 2013.** The Editorial Panel will select four to six choices that will be voted on by *MTMS* readers in September 2013. The most popular name will be presented to the NCTM Board of Directors. Those who submitted the four to six nominated names will each win a $50 gift certificate to the NCTM bookstore.

The individual who submits the winning entry will receive an iPad® and will be featured in the first issue of the newly named journal in August 2014.

MY FAVORITE lesson

Marshall B. Lassak

Introducing Factorials

One of my favorite lessons, inspired by *Anno's Mysterious Multiplying Jar* (1999), introduces the meaning of *factorial* and gives students a sense of the magnitude of factorial numbers.

To begin, I ask a student to draw an outline of a large island on the chalkboard. A second student splits the island into 2 countries, a third student draws 3 mountain ranges in each country, and a fourth student draws 4 kingdoms on each mountain range. At this point, some students comment that drawing the requested items is taking a lot longer than before. To finish the first part of the lesson, I ask a student to represent in some way the 5 villages that exist within each kingdom.

I then pose two questions:

1. If each village has 6 houses, how many houses would students need to draw? Solve this problem in two ways.
2. If drawing each house takes 1 second, how long would drawing all the houses take?

Students are asked to determine the answer to the first question in two ways because some might have prior experience with factorials and I want them to consider another way to solve the problem. The second question evokes a sense of the magnitude of the number being calculated.

We engage in a valuable discussion as we examine the variety of methods for calculating 720 as the number of houses to draw. The sharing of responses provides opportunities to see how different interpretations allow for different sets of calculations that all lead to the correct result. Ultimately, the different products that were calculated are seen to be equivalent to the following expression:

$$6\frac{\text{houses}}{\text{village}} \cdot 5\frac{\text{villages}}{\text{kingdom}} \cdot 4\frac{\text{kingdoms}}{\text{mountain}} \cdot 3\frac{\text{mountains}}{\text{country}} \cdot 2\frac{\text{countries}}{\text{island}} \cdot 1 \text{ island}$$

The emphasis on units is important because it shows how the numbers connect to the given context and gives us a reason to include 1 in the calculation. Converting the time from 720 seconds to 12 minutes gives a more meaningful answer for the students.

To follow up on this part of the lesson, I ask students to determine how many rooms they would have to draw if each house had 7 rooms. Students are also asked to calculate the time it would take to draw all the rooms, still assuming a drawing time of 1 second per item. These questions provide me with feedback on what misunderstandings about the calculations still exist. Most students make the connection to our previous work and compute

$$7\frac{\text{rooms}}{\text{house}} \cdot 720\frac{\text{houses}}{\text{island}}$$

to find that there would be 5,040 rooms, which would take 84 minutes to draw. If students indicate disbelief that it could take this long, I suggest using 1/2 second as a drawing unit of time. Even with a new shorter time unit, the large number of items to draw still takes a significant amount of time to complete. Regardless of the time unit, students recognize the magnitude of the number.

My final two questions require students to determine how many pieces of furniture would need to be drawn if each room in the house had 8 pieces of furniture and, of course, to then calculate how long it would take to draw all the furniture using the new time unit of 1/2 second per item. These questions lead to the payoff moment—when a student exclaims that it will take more than 5 hours to draw 40,320 pieces of furniture!

The lesson closes with the introduction of factorial notation. I explain that 8! means the same thing as $8 \cdot 7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. With the lesson experience, students are able to give meaning to *factorial* on the basis of the context. In addition, they further develop number sense for the magnitude of the numbers being calculated.

REFERENCE

Anno, Masaichiro, and Mitsumasa Anno. 1999. *Anno's Mysterious Multiplying Jar*. New York: Penguin Putnam Books for Young Readers.

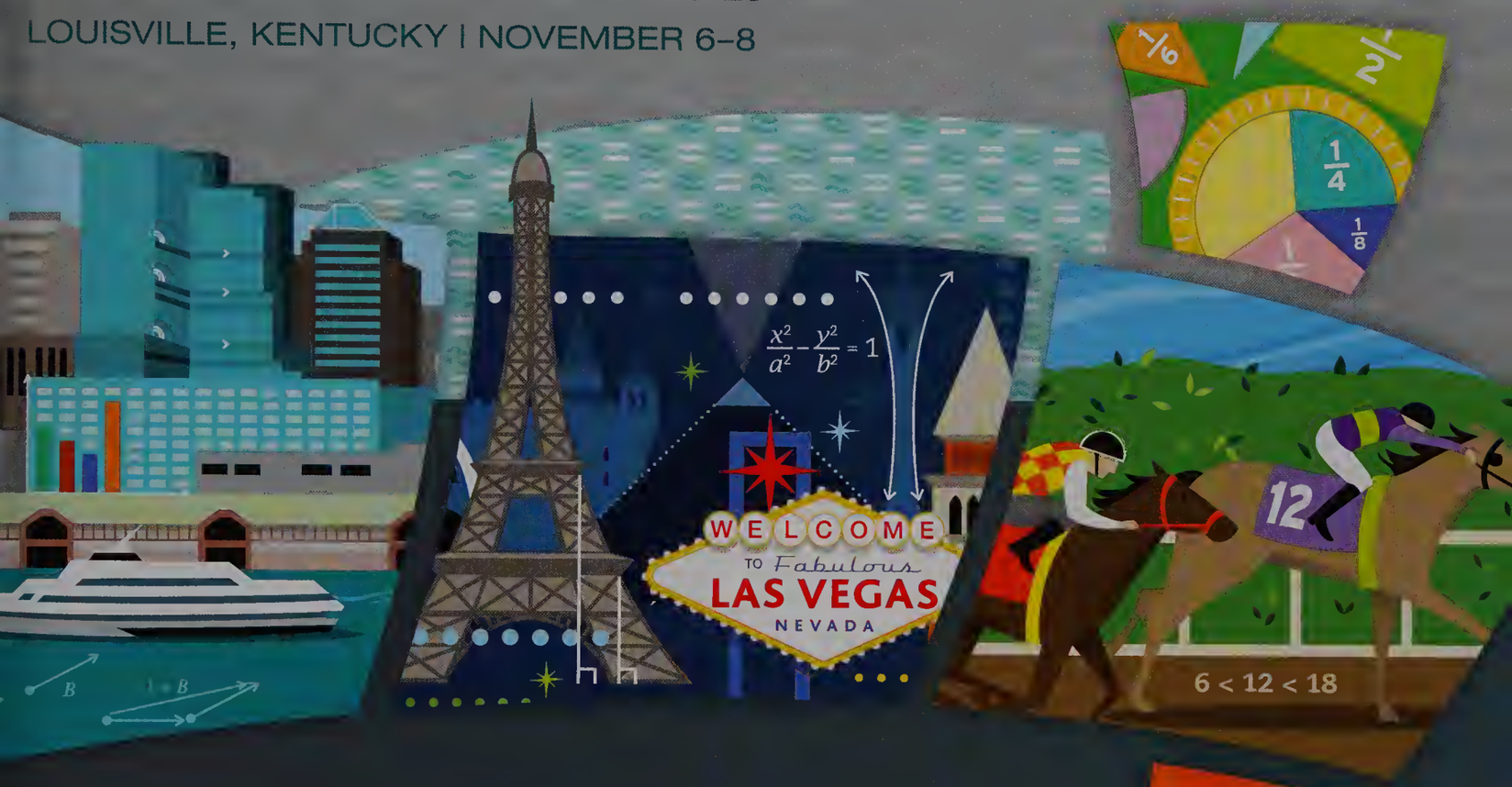
The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

MARSHALL B. LASSAK, mblassak@eiu.edu, works with current and future mathematics teachers, both elementary school and secondary school, at Eastern Illinois University in Charleston. He is interested in effectively using literature as a context for learning mathematics.

NCTM 2013 Regional Conferences & Expositions

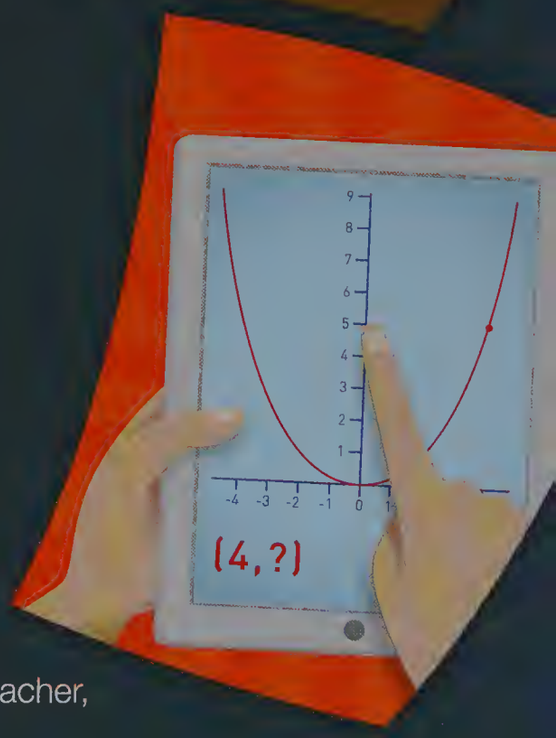
BALTIMORE, MARYLAND | OCTOBER 16–18
LAS VEGAS, NEVADA | OCTOBER 23–25
LOUISVILLE, KENTUCKY | NOVEMBER 6–8



Help Your Students Succeed in a Competitive World

In a global society with rapidly changing technology your students need the right tools to succeed. So take the next step to help them grow—focus on the latest topics for math education at an NCTM Regional Conference. By attending you and your colleagues will:

- Learn more about the latest **technology**
- Discover new and effective **intervention** methods
- Refine your **assessment** techniques
- And more!



Whether you're a classroom teacher, coach, administrator, preservice teacher, or math specialist, this conference has something for you.

NCTM® | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
(800) 235-7566 | WWW.NCTM.ORG

Visit **www.nctm.org/meetings** for up-to-date information.

3 Levels

of professional development offered at beginning, intermediate and advanced levels of experience

2 Products

to choose from when you register:
TI-Nspire™ technology or the TI-84 Plus family

1 Valuable Opportunity

to learn how to use TI technology to effectively engage students in your math or science classroom

T³™ Summer Workshops

T³ Summer Workshops offer hands-on, subject-specific technology professional development for new and experienced educators on key education topics and on how to integrate technology effectively into the math and science classroom.

Led by renowned T³ instructors, the T³ Summer Workshops are an exceptional value. Included in the registration fee are a TI-Nspire™ CX handheld and TI-Nspire™ Teacher Software or a TI-84 Plus C Silver Edition graphing calculator and TI SmartView™ Software.

Find a list of workshop dates and locations, view course descriptions and register now at: **education.ti.com/go/t3summer**.

Register today!

TEXAS INSTRUMENTS

©2013 Texas Instruments AD1119